VOLUME 35, NUMBER 3, SEPTEMBER 2018

ALGEBRAS, GROUPS AND GEOMETRIES Volume 35, Number 3, September 2018

ALGEBRAS, GROUPS AND GEOMETRIES

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ALGEBRAS GROUPS AND GEOMETRIES

VOLUME 35, NUMBER 3, SEPTEMBER 2018

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DOI 10.29083/AGG.35.03.2018

ON SURFACE MEASURES OF CONVEX BODIES AND GENERALIZATIONS OF KNOWN TANGENTIAL IDENTITIES

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Received November 20, 2019
Revised December 3, 2019

Abstract

One theme of this paper is to extend known results from polygons and balls to the general convex bodies in $n-$ dimensions. An another theme stems from approximating a convex surface with a polytope surface. Our result gives a sufficient and necessary condition for an natural approximation method to succeed (in principle) in the case of surfaces of convex bodies. Thus, Schwartz's paradox does not affect our method. This allows us to define certain surface measures on surfaces of convex bodies in a novel and simple way.

Keywords: Convex Geometry, Polytope Approximation, Surface Measures

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1 Introduction

Archimedes discovered the 2-dimensional formula for a disk $D(x, r)$

$$\frac{|A|}{|\partial A|} = \frac{1}{2}R$$

In the early millenia Apostol and Mnatsakanian extended above equality to the class of polygons that they called "tangential" [1]. We will prove that if we extend the definition on tangentiality we obtain in n - dimensions (for that class of polytopes)

$$\frac{|A|}{|\partial A|} = \frac{1}{n}R,$$

where again R is the radius of the inscribed maximal ball.
We will extend above results to general convex bodies. This is achieved via defining a surface measure on convex bodies that we will call a normal measure. In addition we will show that a convex body can be defined as a countable intersection of half-spaces.

2 Some definitions and known results

A convex body is a compact and convex set. We expect always that $0 \in K$. For any measurable set $K \in \mathbf{R}^n$ we use $|K|$ as sign of the Lebesgue measure of that set. For sets $A, B \in \mathbf{R}^n$ we define the Minkowski addition as

$$A + B = \{a + b : a \in A \quad b \in B\}.$$

We define the set dilation as

$$\lambda A = \{\lambda a : a \in A\}.$$

We always use B_2^n an unit ball in n dimensions. The surface area of an convex body K is defined as

$$|\partial(K)| := \lim_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon} \tag{1}$$

We use H_θ for the $n-1$ dimensional hyperplane going through the origo and orthogonal to $\theta \in S^n$. In our sense a polyhedra is an intersection of finite number of closed half-spaces containing origo. A polytype is an polyhedra with a finite Lebesque measure. See for example [5] and references therein. For convex bodies in John's position, the inradius R is defined as the radius of the John's ball [2]. We need the following theorem.

Theorem 1. *If (X, d) is a compact metric space, then $(P(X), dP)$ is a compact metric space, where dp is the Prokhorov metric of weak convergence.*

A proof can be found in [3].

3 On polytopes and normal points

A convex body can be defined as a compact set

$$K = \bigcap_{i \in I} H',$$

where each H' is the half space containing origo that is bounded by the corresponding hyperplane H [5]. It can be proved that in each $n-1$-dimensional hyperplane H there exists an unique point x with the vector $\mathbf{x}$ orthogonal to the hyperplane H [5]. These points x_i are the unique solutions to the minimization problems

$$\min_{y \in H} ||y|| = ||x_i||,$$

where for each hyperplane $y \in H$, and H is the boundary of a half space used to define K. We call these points normal points. They depend on the position of K. They exists as long as the origo belongs to the interior of K. We prove that the convex body inherits the normal points from the half spaces used to define it.

Theorem 2. *Let K be a convex set containing origo. If the half-space H is used in the definition of K, then the normal point of H belongs $K \cap H$.*

Proof. If x_i is defined as the solution to

$$\langle y, x_i \rangle = 0,$$

then a closed origo centered ball of radius $||\mathbf{x}_i||$ contains only that point from the hyperplane H, and because the ball has that common supporting hyperplane with K, we have the claim. □

In light of the theorem the next corollary is clear.

Corollary 1. *Let P be a polytope. Then each facet of P contains a normal point and the number of normal points is less than the number of facets.*

We define minimal normal points as follows. They are the normal points of minimal value $||\cdot||_2$ norm. We will use the following definition.

Definition 1. *A convex body is tangential iff every normal point is minimal.*

With respect to any normal point x_i we denote F_i as the corresponding facet. That is the facet that is orthogonal to $\mathbf{x_i}$ and containing x_i. For each K we define the Borel space $(K, \mathcal{X})$. We consider the finite intersection of half spaces used to define K. For each intersection $\bigcap_{i=1}^{m} H'$ of half spaces forming a polytope and used to define K we define a probability measure

$$\mu_j := \frac{\sum_{i=1}^{m} |F_i| \delta_{x_i}}{\sum_{i=1}^{m} |F_i|},$$

where every δ_{x_i} is a Dirac measure.

Theorem 3. *Let $x_1, ..., x_m$ be the normal points of a polytope P, then*

$$\frac{1}{n}\mathbf{E}(||x||) = \frac{|A|}{|\partial A|},$$

where $||x||$ is the discrete random variable obtaining values $||x_1||, .., ||x_m||$ with probabilities $|F_1|, ..., |F_m|$, respectively.

Proof. We know that a volume of an n-dimensional cone A_i is given by

$$|A_i| = \frac{|\text{base}| * \text{height}}{n}.$$

The theorem 2 implies that the polytope P's volume is

$$|P| = \frac{1}{n}\sum_{i=1}^{m} ||x_i|||F_i|,$$

where F_i is the facet corresponding to the normal point x_i. Thus,

$$\frac{|P|}{|\partial P|} = \frac{\sum_{i=1}^{m} ||x_i|||F_i|}{n\sum_{i=1}^{m}|F_i|} = \frac{1}{n}\mathbb{E}(||x||).$$

□

Corollary 2. *Let a polytope P be tangential, then*

$$\frac{1}{n}R = \frac{|A|}{|\partial A|},$$

where R is the inradius of K.

Naturally we wan't to extend the theorem and it's corollary to the general convex bodies. We will first do that for the corollary. For that we need a lemma.

Lemma 1. *For any point $x \in \partial K$, there exists a normal point x' s.t $x \in H_{x'}$.*

Proof. Now, x belongs to some supporting hyperplane. To this supporting hyperplane there corresponds at least one normal point and via theorem 2 the normal point belongs to ∂K. □

Next we proof that the space of normal points N is separable as a metric compact space.

Theorem 4. *The space of normal points is closed in the subspace topology and the space is separable.*

Proof. Suppose that there exists a Cauchy sequence of normal points (x_j) s.t

$$\lim_{j=1} x_j = x.$$

It holds that $x \in \partial K$ because the surface is closed. Thus, via lemma 1 it holds that $x \in H_{x'}$, where x' is a normal point. Now, if $y \in H_{x'}$ it holds that

$$\lim_{j\to\infty} \langle x_j, x' - y\rangle = \langle x, x' - y\rangle = 0$$

because of the continuity of the inner product. Thus, $H_{x'}$ is orthogonal to $\mathbf{x}$ and $x' = x$. In addition it's well known that every compact metric space is separable [10]. □

For any sequence of normal points (x_j) we denote the corresponding sequence of nested polyhedras to be the sequence

$$(\cap_{j=1}^{m} H_{x_j}) := (P_m).$$

The following theorem is a key theorem.

Theorem 5. *Let (x_j) be any sequence of normal points. Then for the corresponding nested sequence of polyhedras (P_m) it holds that*

$$\bigcap_{m=1}^{\infty} P_m = K,$$

iff the set of the corresponding normal points is dense in the space of normal points of K.

Proof. Let the set of normal points corresponding to the sequence (x_j) be dense in N. Because $\bigcap_{m=1}^{\infty} P_m$ is some convex body, the set of it's normal points is closed. Thus, it follows from the definition of K and from theorem 2 that $K = \bigcap_{m=1}^{\infty} P_m$. On the other hand, if the sequence of normal points is not dense in N, then there exist a normal point of K and a halfspace used in the definition of K that has a boundary that is not a supporting hyperplane to $\bigcap_{m=1}^{\infty} P_m$. Thus,

$$\bigcap_{i=1}^{\infty} P_m \neq K.$$

□

We proved that the space of normal points is separable so it follows from 5 that we can define polytope as an countable intersection of suitable half-spaces. Next we prove a corollary to 5.

Corollary 3. *It holds that*

$$\lim_{m\to\infty} |P_m| = |K|$$

iff the set of normal points corresponding to the polyhedras is dense in N.

Proof. We need only to prove the following. If the set of normal points is not dense then

$$K| < |\bigcap_{i=1}^{\infty} P_m|.$$

Via the theorem 5 the sets are not the same. So there exists $x \in \bigcap_{i=1}^{\infty} P_m \cap K^{\complement}$. Now,

$$\min_{y_i \in K} ||x - y_i|| = ||x - y|| > 0,$$

because there exists a hyperplane separating K and x. It follows that

$$|K| < |\mathrm{conv}(x, K)| \leq |\bigcap_{i=1}^{\infty} P_m|,$$

because there exists a small ball $B(x, \epsilon) \subset \bigcap_{i=1}^{\infty} P_m \cap K^{\complement}$. □

4 On normal measures on convex bodies

In this section we will define a surface measure on convex surfaces in $n-$ dimensions. In general the measure is not continuous, because it can be discrete in the segments of a convex body (for example on the ends of a cylinder). First we will prove the existence of the measure. We use the sequences of measures defined in the last section and we will prove that the measures convergence strongly to a measure.

Lemma 2. *For each Borel set $A \subset K$, the sequence of $(u_m(A))$ is a Cauchy sequence.*

Proof. Let the Borel set A be fixed. W.L.O.G we can assume that there exist m such that

$$\mu_m(A) > 0. \tag{2}$$

Let us denote

$$\mu_m(A) = \frac{|\partial P'_m|}{|\partial P_m|}$$

and

$$\mu_{m+1}(A) = \frac{|\partial P'_{m+i}|}{|\partial P_{m+1}|},$$

where $\partial P'_m$ and $\partial P'_{m+1}$ are the unions of those facets that contain the normal points of $A \cap P_m$ and $A \cap P_{m+1}$, respectively. Now, let us define

$$B' := P'_m - P'_{m+1}.$$

If $P'_m = P'_{m+1} \cup B'$ it can be proved (for example considering that the $n-1$ dimensional Hausdorff-measure of the facets agree with the definition (1)) that

$$0 < |\partial P'_m| = |\partial(P'_{m+1} \cup B')| = |\partial P'_{m+1}| + |\partial B'| - |\partial(P'_{m+1} \cap B')|, \tag{3}$$

where the first inequality above follows from our asumption (2). It follows that B' is not the empty set. So we have

$$|\partial(P'_{m+1} \cap B')| = |\partial P'_{m+1} \cap \partial B'|.$$

Thus, we can rewrite (3) as

$$0 < |\partial P'_m| = |\partial P'_{m+1}| + |\partial B'| - |\partial P'_{m+1} \cap \partial B'|,$$

It follows that

$$|\partial P'_{m+1}| \leq |\partial P'_m|.$$

Thus $|\partial P'_m|$ converges to, say, $|K'|$. It follows that

$$u_m(A) = \frac{|\partial P'_m|}{|\partial P_m|}$$

converges. □

The total variation distance of probability measures is defined as

$$||\mu - v||_{TV} := 2 \sup_A |\mu(A) - v(A)|.$$

Above is an stronger convergence than

$$\lim_{m \to \infty} \mu_m(A) = \mu(A),$$

for any Borel set $A \subset K$.

Theorem 6. *The set function defined by* $\lim_{j \to \infty} u_j = \mu$ *is a probability measure on* $(K, \mathcal{X})$, *where the metric is understood to be the Prokhorov metric of weak convergence.*

Proof. By lemma 2 $\lim_{j \to \infty} u_j(A) \geq 0$ is a Cauchy sequence. Thus, there is a set function μ such that $\lim_{j \to \infty} u_j = \mu$. We just proved that μ_j converges strongly to μ. Because

$$||\mu_i - \mu_j||_{TV} := 2 \sup_A |\mu_i(A) - \mu_j(A)| \leq 2|K|$$

is uniformly bounded and the space is Polish, the strong convergence implies the weak convergence. Thus μ is a probability measure via the theorem 1. □

We need the following theorem.

Theorem 7. *For any measurable function* f *on* K *it holds that*

$$\int_K f\mu = \lim_{m \to \infty} \int_K f\mu_m. \tag{4}$$

Proof. The claim follows straight from the lemma for Borel sets. Thus, it follows for simple functions. If f is a nonnegative measurable function then there exists a monotonically growing sequence of simple functions such that

$$\int_K f\mu - \sum_{i=1}^{n} \alpha_i \mu(A_i)| < \epsilon.$$

Thus,

$$|\int_K f\mu + \lim_{m\to\infty}\sum_{i=1}^{n}\alpha_i\mu_m(A_i)| < \epsilon$$

for all $\epsilon > 0$ and we have the claim.

□

We need to prove that the measure does not depend on the sequence of polytopes, if the sequence of normal points is dense.

Theorem 8. *If A is Borel set it holds that*

$$\lim_{j\to\infty}\int_K 1_A\mu_j = \int_K 1_A\mu = \lim_{l\to\infty}\int_K 1_A v_l = \int_K 1_A v.$$

Proof. If

$$v(A) < \mu(A),$$

then there exists k s.t

$$v_l(A) < \mu(A),$$

when $l > k$. Thus via previous theorem 7 this is a contradiction.

□

For each convex body K we call the measure just defined the normal measure of K.

Next we generalize the result 3 from polytopes to general convex bodies.

Theorem 9. *For any convex body it holds that*

$$\frac{1}{n}\mathbb{E}||x|| = \frac{|K|}{|\partial K|},$$

where $x \in N$ are the normal points of K and the expected value is understood with respect to the measure given by the theorem 6.

Proof. Now, via theorem 7

$$\lim_{j\to\infty}\int 1_A\mu_j \geq \int 1_A\mu$$

for all open sets. It follows that

$$\mathbb{E}(|f|)\mu_j \to \mathbb{E}(|f|)\mu$$

for all bounded continuous functions.

□

The tangential case is an easy corollary.

Corollary 4. *For tangential convex body K, it holds that*

$$\frac{R}{n} = \frac{|K|}{|\partial K|},$$

where R is the inradius of K.

The next theorem is an analogue of the corollary 3 from the last section.

Theorem 10. *It holds that*

$$\lim_{m\to\infty} |\partial P_m| = |\partial K|$$

iff the set of normal points corresponding to the polyhedras is dense in N.

Proof. Via the theorem 5 we need only to prove that if

$$|\partial K| = |\partial \bigcap_{i=1}^{\infty} P_m|,$$

then the set of normal points is dense. Now, if it holds that

$$|K| < |\bigcap_{i=1}^{\infty} P_m|.$$

It follows that

$$|\partial(\bigcap_{i=1}^{\infty} P_m)| - |\partial K|$$
$$= \lim_{\epsilon\to 0} \frac{|\bigcap_{i=1}^{\infty} P_m + \epsilon B_2^n| - |\bigcap_{i=1}^{\infty} P_m| - |K + \epsilon B_2^n| + |K|}{\epsilon} < 0.$$

which is a contradiction.

□

References

[1] T.M. Apostol and M.A. Mnatsakanian, *Figures Circumscribing Circles*, American Mathematical Monthly, December 2004, 853–863.

[2] K.Ball, *An Elementary Introduction to Modern Convex Geometry*, Flavors of Geometry, MSRI Publications, Volume 31, 1997.

[3] O.Gaans, Probability measures on metric spaces, available at https://www.math.leidenuniv.nl/ vangaans/jancol1.pdf

[4] A.Giannopoulos, *Notes on isotropic convex bodies*, Warsaw, October 2003.

[5] Daniel Hug, Wolfgang Weil, *A Course on Convex Geometry*, available at
https://www.fmf.uni-lj.si/ lavric/hug&weil.pdf

[6] Frieda James, *Surface Area and The Cylinder Area Paradox*, available at
https://www.maa.org/sites/default/files/
pdf/upload_library/22/Polya/00494925.di020678.02p0385w.pdf

[7] P. Billingsley, *Convergence of Probability Measures*, John Wiley and Sons Inc, New York, 1999.

[8] T. Rado *The isoperimetric inequality and the Lebesgue definition of surface area*, Transactions of the American Mathematical Society, 1947.

[9] W. Rudin *Real and Complex Analysis*, McGraw-Hill, New York, 1970.

[10] J.Munkres *Topology*, Prentice Hall Inc, New Jersey, 2000, 1975.

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DOI 10.29083/AGG.35.03.2018

ZUR THEORIE DER q, ω-LIESCHEN MATRIXGRUPPEN

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Received April 29, 2019

Abstract

Based on the three papers by Hahn 1949, Annaby et. al. 2012 and Varma et. al. 2018, we introduce the matrix of multiplicative q, ω-polynomials of order $M \in \mathbb{Z}$ with the corresponding q-addition. We prove that this constitutes a so-called q, ω-Lie group with two dual q, ω-multiplications. We also show that the corresponding q, ω-Bernoulli and q, ω-Euler matrices form q, ω-Lie subgroups. In the limit $\omega \to 0$ we obtain corresponding formulas for q-Appell polynomial matrices.

Primary 17B37; Secondary 11B68, 33D15
Keywords— q, ω-Lie group; multiplicative q-Appell polynomial matrix; Hahn–Pascal matrix

ZUSAMMENFASSUNG. Basierend auf den drei Veröffentlichungen von Hahn 1949, Annaby et. al. 2012 und Varma et. al. 2018, führen wir die multiplikative q, ω-Polynommatrix der Ordnung $M \in \mathbb{Z}$ ein, mit der entsprechenden q-Addition. Wir beweisen, dass dies eine sogenannte q, ω-Liesche Gruppe mit zwei dualen q, ω-Multiplikationen darstellt. Wir zeigen auch, dass die entsprechenden q, ω-Bernoulli und q, ω-Euler Matrizen q, ω-Liesche Untergruppen bilden. Im Grenzwert $\omega \to 0$ erhalten wir entsprechende Formeln für q-Appell-Polynommatrizen.

INHALTSVERZEICHNIS

1. Einführung
2. Die q, ω-Liesche Gruppe von q, ω-Appellschen Polynommatrizen
3. Der Matrixansatz
3.1. Multiplikative q, ω-Appellsche Polynommatrizen
3.2. q, ω-Bernoulli und q, ω-Eulersche Polynome
4. Schlussfolgerung
Literatur

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1. Einführung

Wir stellen einige neue Konzepte für q, ω Polynommatrizen vor, von denen einige vorher nur im q-Fall aus den Artikeln des Autors bekannt waren. Durch die logarithmische Methode für q-Analysis erfolgt dieser Übergang fast automatisch, weil die Addition durch die q, ω-Addition ersetzt wird. In dem Artikel [8] wurden Matrixgruppen mit zwei dualen Multiplikationen eingeführt. Später in [10] wurde bewiesen, dass die sogenannte q-Appell-Polynommatrix-Gruppe ein erstes konkretes Beispiel von q-Lie-Gruppen war. Obwohl wir die q, ω-Addition verwenden, werden die q-Binomialkoeffizienten beibehalten. Stattdessen wird die Potenz von x zu den zwei Hauptfolgen geändert.

In diesem Artikel werden die vorherigen Formeln mit q-Pascal-Matrizen einfach zu sogenannten q, ω-Pascal-Matrizen erweitert. Summenformeln mit der neuen q, ω-Addition können dabei in Matrixform umgeschrieben werden.

Dieser Artikel ist wie folgt organisiert: Im Abschnitt 1 werden die Hauptdefinitionen angegeben.

Der Hauptzweck des Abschnitts 2 ist die Einführung der q, ω-Addition und der multiplikativen q, ω-Appellschen Polynome und Zahlen. Das Umbral-Kalkül wird immer implizit angenommen. Im Abschnitt 3 werden die relevanten Matrizen und die Hauptmatrix q, ω-Differenzengleichung eingeführt. Im Unterabschnitt 3.1 werden die multiplikativen q, ω-Polynommatrizen zur Vorbereitung für die q, ω-Liesche Gruppe, den Hauptzweck dieses Artikels, eingeführt.

Im Unterabschnitt 3.2 wiederholen wir zunächst die Matrixformen der q, ω-Bernoulli und q, ω-Eulerschen Polynome aus [9] zur Vorbereitung für die Berechnung ihrer Zahleninverse.

Sei $\omega \in \mathbb{R}$, $\omega > 0$. Man setze $\omega_0 \equiv \frac{\omega}{1-q}$, $0 < q < 1$. Sei I ein Intervall, das ω_0 enthält. Wir gehen davon aus, dass die Variable x zu I gehört.

Definition 1. Der Endomorphismus ϵ im Vektorraum der Polynome wird definiert durch

$$\epsilon f(x) \equiv f(qx + \omega). \tag{1}$$

Dieser Endomorphismus ist eine Verallgemeinerung des Operators mit demselben Namen im q-Kalkül [5]. In [3, S. 136] ist bewiesen, dass

$$\epsilon^k f(x) = f(q^k x + \omega\{k\}_q),\ k \in \mathbb{N}. \tag{2}$$

Definition 2. [11] Sei φ eine stetige reelle Funktion von x. Wir definieren den q,ω-Differenzenoperator $\mathrm{D}_{q,\omega}$ wie folgt:

$$\mathrm{D}_{q,\omega}(\varphi)(x) \equiv \begin{cases} \frac{\varphi(qx+\omega)-\varphi(x)}{(q-1)x+\omega}, & \text{if } x \neq \omega_0; \\ \frac{d\varphi}{dx}(x) & \text{if } x = \omega_0. \end{cases} \tag{3}$$

Eine Funktion $f(x)$ ist n Mal q,ω-differenzierbar, wenn $\mathrm{D}^n_{q,\omega} f(x)$ vorhanden ist. Wenn wir darauf hinweisen möchten, dass dieser Operator auf der Variable x operiert, werden wir $\mathrm{D}_{q,\omega,x}$ für den Operator schreiben. Weiterhin, $\mathrm{D}_{q,\omega}(K) = 0$, wie für die Ableitung.

Dieser Operator interpoliert zwischen zwei bekannten Operatoren, dem Nørlundschen Differenzenoperator

$$\triangle_\omega[f(x)] \equiv \frac{f(x+\omega) - f(x)}{\omega}, \tag{4}$$

und der Jacksonschen q-Ableitung

$$(\mathrm{D}_q\varphi)(x) \equiv \begin{cases} \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q \in \mathbb{C}\backslash\{1\},\ x \neq 0; \\ \frac{d\varphi}{dx}(x) & \text{if } q = 1; \end{cases} \tag{5}$$

Die folgende Definition erscheint zum ersten Mal.

Definition 3. Ein q,ω-Analogon des mathematischen Objekts G ist eine mathematische Funktion $F(q,\omega)$ mit der Eigenschaft $\lim_{\omega\to 0} F(q,\omega) = G_q$, das q-Analogon von G. Sowohl F als auch G können Funktionen von mehreren Variablen sein. Sie können auch Operatoren sein. Die Funktion $F(q,\omega)$ wird ω-Analogon von G_q genannt.

Satz 1.1. [3, (16), S. 137] *Der q,ω-Differenzenoperator für ein Produkt von Funktionen.*

$$\mathrm{D}_{q,\omega}(fg)(x) = \mathrm{D}_{q,\omega}(f)(x)g(x) + f(qx+\omega)\mathrm{D}_{q,\omega}(g)(x). \tag{6}$$

Bemerkung 1. Diese Formel wird zum Nachweis von (28) verwendet.

Wir führen nun zwei Hauptfolgen ein, die die Ciglerschen Polynome in [5, 5.5] verallgemeinern.

Definition 4.

$$[13,\ (15)]\quad [x]_{q,\omega}^{k} \equiv \prod_{m=0}^{k-1}(q^m x + \omega\{m\}_q). \tag{7}$$

$$[13,\ (16)]\quad (x)_{q,\omega}^{k} \equiv \prod_{m=0}^{k-1}(x - \omega\{m\}_q), \tag{8}$$

wobei $\{m\}_q$ das q-Analogon von m bezeichnet.

Die beiden folgenden Formeln entsprechen der Formel $\mathrm{D}x^n = nx^{n-1}$:

$$[12,\ 2.5],\ [13,\ (17)]\quad \mathrm{D}_{q,\omega}(x)_{q,\omega}^{n} = \{n\}_q (x)_{q,\omega}^{n-1}. \tag{9}$$

$$[13,\ (18)]\quad \mathrm{D}_{q,\omega}[x]_{q,\omega}^{n} = \{n\}_q [qx + \omega]_{q,\omega}^{n-1}. \tag{10}$$

Als nächstes führen wir zwei q,ω-Analoga der Exponentialfunktion ein:

Definition 5. Die q,ω-Exponentialfunktion $\mathrm{E}_{q,\omega}(z)$ [13, (21)] wird definiert durch

$$\mathrm{E}_{q,\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{(z)_{q,\omega}^{k}}{\{k\}_q!},\ |(1-q)z - \omega| < 1. \tag{11}$$

Die komplementäre q,ω-Exponentialfunktion $\mathrm{E}_{\frac{1}{q},\omega}(z)$ [13, (26)] wird definiert durch

$$\mathrm{E}_{\frac{1}{q},\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{[z]_{q,\omega}^{k}}{\{k\}_q!},\ |\omega| < 1. \tag{12}$$

Wir haben den Namen geändert zu $\mathrm{E}_{\frac{1}{q},\omega}(z)$, weil $\mathrm{E}_{\frac{1}{q},0}(z) = \mathrm{E}_{\frac{1}{q}}(z)$ [5].

Satz 1.2. [13, (19)] *Die q,ω-Exponentialfunktion ist die einzigartige Lösung der q,ω-Differenzengleichung*

$$\mathrm{D}_{q,\omega} f(z) = f(z),\ f(0) = 1. \tag{13}$$

[13, (24)] *Die komplementäre q,ω-Exponentialfunktion ist die einzigartige Lösung der q,ω-Differenzengleichung*

$$\mathrm{D}_{q,\omega} f(z) = f(qz + \omega),\ f(0) = 1. \tag{14}$$

Satz 1.3. [13, (21)] *Die meromorphe Fortsetzung der q,ω-Exponentialfunktion $\mathrm{E}_{q,\omega}(z)$ ist gegeben durch*

$$\mathrm{E}_{q,\omega}(z) = \frac{(-\omega;q)_\infty}{((1-q)z-\omega;q)_\infty}. \tag{15}$$

[13, (26)] *Die meromorphe Fortsetzung der komplementären q,ω-Exponentialfunktion $\mathrm{E}_{\frac{1}{q},\omega}(z)$ ist gegeben durch*

$$\mathrm{E}_{\frac{1}{q},\omega}(z) = \frac{((q-1)z+\omega;q)_\infty}{(\omega;q)_\infty}. \tag{16}$$

Korollarium 1.4.

$$\mathrm{E}_{q,\omega}(z)\mathrm{E}_{\frac{1}{q},-\omega}(-z) = 1 \tag{17}$$

2. Die q,ω-Liesche Gruppe von q,ω-Appellschen Polynommatrizen

Wir erweitern zunächst einige Definitionen von [8].

Definition 6. Eine q,ω-Liesche Gruppe $(G_{n,q,\omega,\cdot,\cdot_{q,\omega}}, \mathrm{I}_g) \supseteq \mathrm{E}_{q,\omega}(\mathfrak{g}_q)$ ist eine möglicherweise unendliche Menge von Matrizen $\in \mathrm{GL}(n,\mathbb{R})$ und eine Mannigfaltigkeit mit zwei Multiplikationen: $\cdot$, der üblichen Matrixmultiplikation und der verdrehten Matrixmultiplikation $\cdot_{q,\omega}$, die separat definiert wird.

Jede q,ω-Liesche Gruppe hat eine Einheit, die für beide Multiplikationen mit I_g bezeichnet wird. Jedes Element $\Phi \in G_{n,q,\omega}$ hat eine Inverse Φ^{-1} mit der Eigenschaft $\Phi \cdot_{q,\omega} \Phi^{-1} = \mathrm{I}_g$.

Definition 7. Angenommen, $(G_1, \cdot_1, \cdot_{1:q,\omega})$ und $(G_2, \cdot_2, \cdot_{2:q,\omega})$ sind zwei q,ω-Liesche Gruppen, dann ist $(G_1 \times G_2, \cdot, \cdot_{q,\omega})$ eine q,ω-Liesche Gruppe mit dem Namen Produkt-q,ω-Liesche Gruppe. Diese Gruppe hat Gruppenoperationen definiert durch

$$(g_{11}, g_{21}) \cdot (g_{12}, g_{22}) = (g_{11} \cdot_1 g_{12}, g_{21} \cdot_2 g_{22}), \tag{18}$$

und

$$(g_{11}, g_{21}) \cdot_{q,\omega} (g_{12}, g_{22}) = (g_{11} \cdot_{1:q,\omega} g_{12}, g_{21} \cdot_{2:q,\omega} g_{22}). \tag{19}$$

Definition 8. Wenn $(G_{n,q,\omega}, \cdot, \cdot_{q,\omega})$ eine q,ω-Liesche Gruppe ist und $H_{n,q,\omega}$ eine nichtleere Teilmenge von $G_{n,q,\omega}$ ist, dann wird $(H_{n,q,\omega}, \cdot, \cdot_{q,\omega})$ eine q,ω-Liesche Untergruppe von $(G_{n,q,\omega}, \cdot, \cdot_{q,\omega})$ genannt, falls

(1)

(20) $\Phi \cdot \Psi \in H_{n,q,\omega}$ und $\Phi \cdot_{q,\omega} \Psi \in H_{n,q,\omega}$ for all $\Phi, \Psi \in H_{n,q,\omega}$.

(2)

(21) $\Phi^{-1} \in H_{n,q,\omega}$ for all $\Phi \in H_{n,q,\omega}$.

(3) $H_{n,q,\omega}$ eine Untermannigfaltigkeit von $G_{n,q,\omega}$ ist.

Um die folgenden Polynome verwenden zu können, müssen wir die q-Addition verallgemeinern. Die gewöhnliche q-Addition ist der Sonderfall $\omega = 0$. Genau wie bei der q-Addition verwenden wir Buchstaben in einem Alphabet für die q,ω-Additionen. Die Gleichheit der Buchstaben wird mit $\sim$ bezeichnet. Man beachte im Folgenden die Tatsache, dass jeweils die Variable x in $(x)^{\nu}_{q,\omega}$ oder in $[x]^{\nu}_{q,\omega}$ durch die Konstante a multipliziert wird, müssen wir auch ω mit a multiplizieren.

Definition 9. Die NWA q,ω-Addition wird wie folgt definiert:

$$(x \oplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)^{n-k}_{q,\omega} (y)^k_{q,\omega}. \tag{22}$$

Die NWA q,ω-Subtraktion wird wie folgt definiert:

$$(x \ominus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)^{n-k}_{q,\omega} (-y)^k_{q,-\omega}. \tag{23}$$

Die JHC q,ω-Addition wird wie folgt definiert:

$$(x \boxplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)^{n-k}_{q,\omega} [y]^k_{q,\omega}. \tag{24}$$

Die JHC q,ω-Subtraktion wird wie folgt definiert:

$$(x \boxminus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)^{n-k}_{q,\omega} [-y]^k_{q,-\omega}. \tag{25}$$

Korollarium 2.1. *Eine Erweiterung der Formel* [5, 4.29]

$$\mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n = \{n\}_q (x \oplus_{q,\omega} y)^{n-1}, \ \oplus_{q,\omega} \equiv \oplus_{q,\omega} \vee \boxplus_{q,\omega}. \tag{26}$$

Beweis.

$$\text{(27)} \qquad \mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n = \sum_{k=0}^{n-1} \binom{n}{k}_q \{n-k\}_q (x)_{q,\omega}^{n-k-1} (y)_{q,\omega}^k = \mathrm{RS}.$$

□

Satz 2.2. *Die Kettenregel für den q,ω-Differenzenoperator.*

$$\text{(28)} \qquad \mathrm{D}_{q,\omega}\left((ax)_{q,a\omega}^n\right) = a\{n\}_q (ax)_{q,a\omega}^{n-1}.$$

$$\text{(29)} \qquad \mathrm{D}_{q,\omega}\left([ax]_{q,a\omega}^n\right) = a\{n\}_q [aqx + a\omega]_{q,a\omega}^{n-1}.$$

Beweis. Wir beweisen (28) durch Induktion. Die Formel (28) gilt für $n = 1, 2$. Angenommen, sie gilt für $n-1$. Dann haben wir

$$\begin{aligned} &\mathrm{D}_{q,\omega}\left[(ax)_{q,a\omega}^{n-1}(ax - \{n-1\}_q a\omega)\right] \\ \text{(30)} \qquad &\overset{\text{durch(6)}}{=} a(ax)_{q,a\omega}^{n-1} + a^2\left[qx + \omega - \{n-1\}_q\right]\{n-1\}_q (ax)_{q,a\omega}^{n-2} \\ &= a(ax)_{q,a\omega}^{n-1}\left[1 + q\{n-1\}_q\right] = \mathrm{RS}. \end{aligned}$$

Die Formel (29) wird in ähnlicher Weise bewiesen. □

Korollarium 2.3. *Vier q,ω-Additionen für die q,ω-Exponentialfunktion.*

$$\text{(31)} \qquad \mathrm{E}_{q,\omega}(x \oplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,\omega}(y).$$

$$\text{(32)} \qquad \mathrm{E}_{q,\omega}(x \ominus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,-\omega}(-y).$$

$$\text{(33)} \qquad \mathrm{E}_{q,\omega}(x \boxplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},\omega}(y).$$

$$\text{(34)} \qquad \mathrm{E}_{q,\omega}(x \boxminus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},-\omega}(-y).$$

Beweis. Man verwende die Formeln (22) und (24). □

Satz 2.4. *Die q,ω-Differenzen für die q,ω-Exponentialfunktionen sind:*

$$\text{(35)} \qquad \mathrm{D}_{q,\omega}\ \mathrm{E}_{q,a\omega}(ax) = a\ \mathrm{E}_{q,a\omega}(ax),$$

$$\text{(36)} \qquad \mathrm{D}_{q,\omega}\ \mathrm{E}_{\frac{1}{q},a\omega}(ax) = a\ \mathrm{E}_{\frac{1}{q},a\omega}(aqx + a\omega),$$

Beweis. Dies ergibt sich aus der Kettenregel (28) und (29). □

Satz 2.5. *Die NWA q,ω-Addition ist kommutativ und assoziativ.*

Beweis. Ähnlich dem Nachweis für NWA q-Addition. □

Satz 2.6. *Die JHC q,ω-Addition ist assoziativ. Wir nehmen an, dass alle JHC q,ω-Additionen ganz rechts im Funktionsargument stehen.*

Beweis. Dies ergibt sich aus der Assoziativität der Multiplikation. □

Definition 10. Die Ward-ω-Zahl $\overline{n}_{q,\omega}$ wird definiert durch

$$\overline{n}_{q,\omega} \sim 1 \oplus_{q,\omega} 1 \oplus_{q,\omega} \dots \oplus_{q,\omega} 1, \tag{37}$$

wobei die Anzahl der Summanden auf der rechten Seite n ist.

Definition 11. Die Jacksonsche ω-Zahl $\tilde{n}_{q,\omega}$ wird definiert durch

$$\tilde{n}_{q,\omega} \sim 1 \boxplus_{q,\omega} 1 \boxplus_{q,\omega} \dots \boxplus_{q,\omega} 1, \tag{38}$$

wobei die Anzahl der Summanden auf der rechten Seite n ist.

Die allgemeinste Form von Polynom in diesem Artikel ist das q,ω-Appell-Polynom, das wir nun definieren werden.

Definition 12. Sei $\mathcal{A}_{q,\omega}$ die reelle Zahlenfolgen $\{u_{\nu,q}\}_{\nu=0}^{\infty}$, so dass

$$\sum_{\nu=0}^{\infty} |u_{\nu,q}| \frac{r^{\nu}}{\{\nu\}_q!} < \infty, \tag{39}$$

für einen q,ω-abhängiger Konvergenzradius $r = r(q) > 0$, wobei $0 < q < 1$.

Die q,ω-Appellsche Zahlenfolge wird mit $\{\Phi_{\nu,q,\omega}^{(n)}\}_{\nu=0}^{\infty}$ bezeichnet.

Definition 13. Sei $h(t,q,\omega)$, $h(t,q,\omega)^{-1} \in \mathbb{R}[[t]]$. Für $f_n(t,q,\omega) = h(t,q,\omega)^n$ werden die multiplikativen q,ω-Appellschen Zahlen von Grad ν und Ordnung n, $\Phi_{\nu,q,\omega} \in \mathcal{A}_{q,\omega}$ durch die folgende erzeugende Funktion gegeben:

$$f_n(t,q,\omega) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \Phi_{\nu,q,\omega}^{(n)}, \ \Phi_{0,q,\omega}^{(n)} = 1. \tag{40}$$

Der Bequemlichkeit halber fixieren wir den Wert für $n = 0$ und $n = 1$:

$$\Phi_{\nu,q,\omega}^{(0)} \equiv \delta_{0,\nu}, \ \Phi_{\nu,q,\omega}^{(1)} \equiv \Phi_{\nu,q,\omega}. \tag{41}$$

Definition 14. Man vergleiche mit [13, (31)]. Für $f_n(t,q,\omega) \in \mathbb{R}[[t]]$ wie oben wird die multiplikative q,ω-Appellsche Polynomfolge $\{\Phi_{\nu;q,\omega}^{(n)}(x)\}_{\nu=0}^{\infty}$ von Grad ν und Ordnung n durch die folgende erzeugende Funktion gegeben:

$$f_n(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\nu;q,\omega}(x). \tag{42}$$

Definition 15. Der Bequemlichkeit halber fixieren wir wieder den Wert für $n = 0$ und $n = 1$:

$$\Phi^{(0)}_{\nu,q,\omega}(x) = (x)^\nu_{q,\omega},\ \Phi^{(1)}_{\nu,q,\omega}(x) \equiv \Phi_{\nu,q,\omega}(x). \tag{43}$$

Motivation: Die erste Definition folgt, weil die zwei Hauptfolgen die Potenzfunktion ersetzen.

Als nächstes werden Verallgemeinerungen der beiden Formeln [5, 4.107, 4.111] vorgestellt.

Satz 2.7.

$$\mathrm{D}_{q,\omega}\Phi_{\nu;q,\omega}(x) = \{\nu\}_q \Phi_{\nu-1;q,\omega}(x). \tag{44}$$

Die Formel [13, (30)] *in Umbralform:*

$$\Phi_{\nu;q,\omega}(x) \doteq (\Phi_{q,\omega} \oplus_{q,\omega} x)^\nu. \tag{45}$$

Definition 16. Man vergleiche mit [13, (31)]. Für $f_n(t,q,\omega) \in \mathbb{R}[[t]]$ wie oben wird die komplementäre q,ω-Appellsche Polynomfolge $\{\Phi^{(n)}_{\nu;\frac{1}{q},\omega}(x)\}_{\nu=0}^{\infty}$ von Grad ν und Ordnung n durch die folgende erzeugende Funktion definiert:

$$f_n(t,q,\omega)\mathrm{E}_{\frac{1}{q},\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\nu;\frac{1}{q},\omega}(x). \tag{46}$$

Bemerkung 2. Diese Definition wird in der Formel (49) verwendet.

Satz 2.8. *Angenommen, M und K sind die x-Ordnung bzw. y-Ordnung. Dann haben wir ein ω-Analogon von* [10, (43)]*:*

$$\Phi^{(M+K)}_{\nu,q,\omega}(x \oplus_{q,\omega} y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,q,\omega}(y). \tag{47}$$

Beweis. Wir zeigen, dass beide Seiten von (47) dieselbe erzeugende Funktion haben.

$$
\begin{aligned}
& f_{M+K}(t,q,\omega)\mathrm{E}_{q,\omega t}((x\oplus_{q,\omega} y)t) \overset{\text{durch}(22)}{=} f_M(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) \\
(48)\qquad & f_K(t,q,\omega)\mathrm{E}_{q,\omega t}(yt) \overset{\text{durch}(42)}{=} \sum_{k=0}^{\infty}\frac{t^k}{\{k\}_q!}\Phi^{(M)}_{k;q,\omega}(x)\sum_{l=0}^{\infty}\frac{t^l}{\{l\}_q!}\Phi^{(K)}_{l;q,\omega}(y) \\
& =\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\sum_{k=0}^{\nu}\binom{\nu}{k}_q\Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,q,\omega}(y).
\end{aligned}
$$

□

Bemerkung 3. Die Formel (47) definiert $\Phi^{(M+K)}_{\nu,q,\omega}(x\oplus_{q,\omega} y)$ als rechte Seite der Formel. Es gibt keine andere Definition von dieser Funktion.

Satz 2.9. *Angenommen, M und K sind die x-Ordnung bzw. y-Ordnung. Dann haben wir:*

$$
(49)\qquad \Phi^{(M+K)}_{\nu,q,\omega}(x\boxplus_{q,\omega} y)=\sum_{k=0}^{\nu}\binom{\nu}{k}_q\Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,\frac{1}{q},\omega}(y).
$$

Beweis. Wir zeigen, dass beide Seiten von (49) dieselbe erzeugende Funktion haben.

(50)

$$
\begin{aligned}
& f_{M+K}(t,q,\omega)\mathrm{E}_{q,\omega t}((x\boxplus_{q,\omega} y)t) \overset{\text{durch}(24)}{=} f_M(t,q,\omega)\mathrm{E}_{q,\omega t}(xt)f_K(t,\tfrac{1}{q},-\omega) \\
& \mathrm{E}_{\frac{1}{q},\omega t}(yt) \overset{\text{durch}(42),(46)}{=} \sum_{k=0}^{\infty}\frac{t^k}{\{k\}_q!}\Phi^{(M)}_{k;q,\omega}(x)\sum_{l=0}^{\infty}\frac{t^l}{\{l\}_q!}\Phi^{(K)}_{l;\frac{1}{q},\omega}(y) \\
& =\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\sum_{k=0}^{\nu}\binom{\nu}{k}_q\Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,\frac{1}{q},\omega}(y).
\end{aligned}
$$

□

3. Der Matrixansatz

In diesem Abschnitt verallgemeinern wir Resultate aus [9] und [10] durch die Einführung der Variable ω.

Definition 17. [6], [9] Ein q-Analogon der Polya-Veinschen Matrix. Die $n \times n$ Matrix $\mathbf{D}_{n,q}$ ist gegeben durch

$$\begin{aligned} &\mathbf{D}_{n,q}(i, i-1) \equiv \{i\}_q, \ i = 1, \ldots, n-1, \\ &\mathbf{D}_{n,q}(i,j) \equiv 0, \ j \neq i-1. \end{aligned} \tag{51}$$

Die folgende Vektorform für q, ω-Appellsche Polynome und Zahlen wird in den Formeln (69), (89), (90), (108) und (109) verwendet.

$$\phi_{n,q,\omega}(x) \equiv (\Phi_{0,q,\omega}(x), \Phi_{1,q,\omega}(x), \ldots, \Phi_{n-1,q,\omega}(x))^T, \tag{52}$$

$$\phi_{n,q,\omega} \equiv \phi_{n,q,\omega}(0). \tag{53}$$

Definition 18. Die folgende Abkürzung wird verwendet.

$$\xi_{n,q,\omega}(x) \equiv ((x)^0_{q,\omega}, (x)^1_{q,\omega}, (x)^2_{q,\omega}, \ldots, (x)^{n-1}_{q,\omega})^T. \tag{54}$$

Definition 19. Man definiere die q, ω-Appellsche Polynommatrix durch

$$\overline{\Phi}_{n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q \Phi_{i-j,q,\omega}(x), \ 0 \leq i, j \leq n-1. \tag{55}$$

Definition 20. Die q, ω-Appellsche Zahlenmmatrix wird definiert durch

$$\overline{\Phi}_{n,q,\omega}(i,j) \equiv \overline{\Phi}_{n,q,\omega}(0)(i,j), \ 0 \leq i, j \leq n-1. \tag{56}$$

Satz 3.1. *Die Formel (44) kann in Matrixform geschrieben werden. Vergleiche mit* [6, (83)].

$$\mathrm{D}_{q,\omega}\phi_{n,q,\omega}(x) = \mathbf{D}_{n,q}\phi_{n,q,\omega}(x). \tag{57}$$

Angenommen, $y(t)$ ist ein Vektor der Länge n, ist die folgende q, ω-Differenzengleichung in $\mathbb{R}^n$ von grundlegender Bedeutung:

$$\mathrm{D}_{q,\omega}y(t) = \mathbf{D}_{n,q}y(t), y(0) = y_0, \ -\infty < t < \infty. \tag{58}$$

Gemäß der Formel (57), ist die allgemeine Lösung von (58) der q, ω-Appell-Polynomvektor von Grad ν und Ordnung m. Die Anfangswerte sind dann der Vektor von q, ω-Appellschen Zahlen der Ordnung m u.s.w.. Der Anfangswert kann auch die Vektorfunktion e_0 sein. Die Lösung ist dann die Vektorfunktion $\xi_{n,q,\omega}(x)$.

Definition 21. Die q,ω-Pascalsche Matrix $\mathrm{P}_{n,q,\omega}(x)$ ist gegeben durch

$$\mathrm{P}_{n,q,\omega}(i,j)(x) \equiv \begin{cases} \binom{i}{j}_q (x)_{q,\omega}^{i-j}, i \geq j \\ 0, \text{ sonst} \end{cases} \tag{59}$$

Diese Matrix wird in der Formel (71) verwendet.

Satz 3.2. *Die allgemeine Lösung von (58) kann auch geschrieben werden:* $y(t) = \mathrm{E}_{q,\omega}(\mathbf{D}_{n,q}t)y_0$. *Dies ist eigentlich eine endliche Reihe, die sich in der folgenden Form ausdrücken lässt:*

$$\sum_{k=0}^{n-1} \frac{(\mathbf{D}_{n,q}t)_{q,\omega}^k}{\{k\}_q!} \equiv \mathrm{P}_{n,q,\omega}(t). \tag{60}$$

Der folgende Sonderfall wird häufig verwendet.

Definition 22. Die q,ω-Pascalsche Matrix $\mathrm{P}_{n,q,\omega}$ ist gegeben durch

$$\mathrm{P}_{n,q,\omega}(i,j) \equiv \mathrm{P}_{n,q}(i,j)(1) = \binom{i}{j}_q (1)_{q,\omega}^{i-j}, i,j = 0,\ldots,n-1. \tag{61}$$

Des Weiteren haben wir das folgende q,ω-Analogon von [1, S. 233 (7)], was daraus folgt, dass $\mathrm{P}_{n,q,\omega}(t)$ eine q,ω-Exponentialfunktion ist.

$$\mathrm{P}_{n,q,\omega}(s \oplus_{q,\omega} t) = \mathrm{P}_{n,q,\omega}(s)\mathrm{P}_{n,q,\omega}(t), \ s,t \in \mathbb{C}. \tag{62}$$

Das impliziert

$$\mathrm{P}_{n,q,\omega}^k = \mathrm{P}_{n,q,\omega}(\overline{k}_{q,\omega}). \tag{63}$$

Durch (62) erhalten wir viele kombinatorische Identitäten. Einige davon sind

$$\sum_{k=j}^{i} \binom{i}{k}_q \binom{k}{j}_q (1)_{q,\omega}^{i-k}[-1]_{q,\omega}^{k-j} = \delta_{i,j} \tag{64}$$

und

$$\sum_{k=j}^{i} \binom{i}{k}_q \binom{k}{j}_q (1)_{q,\omega}^{i-k}(1)_{q,\omega}^{k-j} = (\overline{2_{q,\omega}})^{i-j} \binom{i}{j}_q, \ i \geq j. \tag{65}$$

3.1. Multiplikative q,ω-Appellsche Polynommatrizen.

Definition 23. Ein ω-Analogon von [10, (47)]. Die multiplikativen q,ω-Appell-Polynommatrizen $(\mathcal{M}_{x,q,\omega})$ mit Elementen $\overline{\Phi}^{(M)}_{n,q,\omega}(x)$ der Ordnung $M \in \mathbb{Z}$ sind definiert durch

$$\overline{\Phi}^{(M)}_{n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q \Phi^{(M)}_{i-j,q,\omega}(x),\ 0 \le i,j \le n-1. \tag{66}$$

Definition 24. Die multiplikativen q,ω-Appell Zahlenmatrizen oder die q,ω-Übertragung-Matrizen $(\mathcal{M}_{q,\omega})$ mit Elementen $\overline{\Phi}^{(M)}_{n,q,\omega}$ der Order $M \in \mathbb{Z}$ sind definiert durch

$$\overline{\Phi}^{(M)}_{n,q,\omega}(i,j) \equiv \overline{\Phi}^{(M)}_{n,q,\omega}(0)(i,j),\ 0 \le i,j \le n-1. \tag{67}$$

Satz 3.3. *Eine Formel für die q,ω-Übertragung-Matrix*

$$\overline{\Phi}_{n,q,\omega} = f_n(t,q,\omega)\mathbf{D}_{n,q}, \tag{68}$$

wobei $f_n(t,q,\omega)$ durch (42) definiert wird.

Beweis. Dies ist ähnlich wie bei der Formel [9, (79)]. □

Satz 3.4. *Der q,ω-Appell-Polynomvektor von x kann als Produkt der q,ω-Appellzahlen-Matrix mal $\xi_{n,q,\omega}(x)$ ausgedrückt werden. Ein q,ω-Analogon von* [2, (3.9), S. 432] *und ein ω-Analogon von* [10, (49)].

$$\phi_{n,q,\omega}(x) = \overline{\Phi}_{n,q,\omega}\xi_{n,q,\omega}(x). \tag{69}$$

Beweis. Dies ist die Formel (45) in Matrixform.

□

Satz 3.5. *Der q,ω-Appell-Polynomvektor von $x \oplus_{q,\omega} y$ kann als Produkt der q,ω-Appellschen Matrix von x mal der q,ω-Appellschen Vektor von y ausgedrückt werden. Ein q,ω-Analogon von* [2, (4.1), S. 436].

$$\phi_{n,q,\omega}(x \oplus_{q,\omega} y) = \overline{\Phi}_{n,q,\omega}(x)\phi_{n,q,\omega}(y). \tag{70}$$

Beweis. Dies ist die Formel (47) in Matrixform. □

Satz 3.6. *Ein q,ω-Analogon von* [2, S. 436].

$$\xi_{n,q,\omega}(x \oplus_q y) = \mathrm{P}_{n,q,\omega}(x)\xi_{n,q,\omega}(y). \tag{71}$$

Beweis. Wir haben durch [12, 2.3]

$$\begin{aligned}(72)\qquad &\xi_{n,q,\omega}(x \oplus_q y)(i) = (x \oplus_q y)^i_{q,\omega} \\ &= \sum_{k=0}^{i} \binom{i}{k}_q (x)^{i-k}_{q,\omega}(y)^k_{q,\omega} \overset{\text{durch}(59)}{=} (\mathrm{P}_{n,q,\omega}(x)\xi_{n,q,\omega}(y))(i).\end{aligned}$$

□

Satz 3.7. *Ein q,ω-Analogon von* [2, S. 436].

$$(73)\qquad \overline{\Phi}^{(M+K)}_{n,q,\omega}\xi_{n,q,\omega}(x \oplus_{q,\omega} y) = \overline{\Phi}^{(K)}_{n,q,\omega}\overline{\Phi}^{(M)}_{n,q,\omega}(x)\xi_{n,q,\omega}(y).$$

Beweis. Die beiden Matrizen $\overline{\Phi}^{(M)}_{n,q,\omega}(x)$ und $\overline{\Phi}^{(K)}_{n,q,\omega}$ sind Potenzreihen in $\mathbf{D}_{n,q}$ und wir haben

$$\begin{aligned}(74)\qquad &\overline{\Phi}^{(M)}_{n,q,\omega}(x)\phi^{(K)}_{n,q,\omega}(y) \\ &\overset{\text{durch}(69)}{=} \overline{\Phi}^{(M)}_{n,q,\omega}(x)\overline{\Phi}^{(K)}_{n,q,\omega}\xi_{n,q,\omega}(y) = \overline{\Phi}^{(K)}_{n,q,\omega}\overline{\Phi}^{(M)}_{n,q,\omega}(x)\xi_{n,q,\omega}(y).\end{aligned}$$

Andererseits gemäß der Formel (47) ist dies gleich

$$(75)\qquad \phi^{(M+K)}_{n,q,\omega}(x \oplus_{q,\omega} y) \overset{\text{durch}(69)}{=} \overline{\Phi}^{(M+K)}_{n,q,\omega}\xi_{n,q,\omega}(x \oplus_{q,\omega} y).$$

Die Formel (73) folgt, indem die letzten Ausdrücke von (74) und (75) gleichgesetzt werden. □

Bemerkung 4. Die Formel (73) ergibt eine implizite Definition der Funktion $\xi_{n,q,\omega}(x \oplus_{q,\omega} y)$.

Satz 3.8. *Ein ω-Analogon von* [10, (55)]. *Wir gehen davon aus, dass M und K die x-Ordnung bzw. y-Ordnung sind. Die Formel (47) kann in der folgenden Matrixform umgeschrieben werden, wobei $\cdot$ auf der rechten Seite eine Matrixmultiplikation bezeichnet.*

$$(76)\qquad \overline{\Phi}^{(M+K)}_{n,q,\omega}(x \oplus_{q,\omega} y) = \overline{\Phi}^{(M)}_{n,q,\omega}(x) \cdot \overline{\Phi}^{(K)}_{n,q,\omega}(y).$$

Beweis. Wir berechnen das Matrixelement (i,j) der Matrixmultiplikation auf der rechten Seite.

$$
\begin{aligned}
&\sum_{k=j}^{i} \binom{i}{k}_q \Phi^{(M)}_{i-k,q,\omega}(x) \binom{k}{j}_q \Phi^{(K)}_{k-j,q,\omega}(y) \\
&= \binom{i}{j}_q \sum_{k=j}^{i} \binom{i-j}{k-j}_q \Phi^{(M)}_{i-k,q,\omega}(x) \Phi^{(K)}_{k-j,q,\omega}(y) \\
&= \binom{i}{j}_q \sum_{k=0}^{i-j} \binom{i-j}{k}_q \Phi^{(M)}_{i-j-k,q,\omega}(x) \Phi^{(K)}_{k,q,\omega}(y) \\
&\overset{\text{durch}(47)}{=} \binom{i}{j}_q \Phi^{(M+K)}_{i-j,q,\omega}(x \oplus_{q,\omega} y) = \mathrm{LS}.
\end{aligned} \tag{77}
$$

□

Durch die Formel (66) sind die $\overline{\Phi}^{(M)}_{n,q}(x)$-Matrizen mit Matrixelementen q,ω-Appellsche Polynome multipliziert mit q-Binomial-Koeffizienten, und wir gelangen zur nächsten wichtigen Definition.

Definition 25. Ein ω-Analogon von [10, (57)]. Wir definieren die zweite Matrixmultiplikation $\cdot_{q,\omega}$ durch

$$
\overline{\Phi}^{(M)}_{n,q,\omega}(x) \cdot_{q,\omega} \overline{\Phi}^{(K)}_{n,q,\omega}(y) \equiv \overline{\Phi}^{(M+K)}_{n,q,\omega}(x \boxplus_{q,\omega} y), \tag{78}
$$

wobei $\overline{\Phi}^{(M+K)}_{n,q,\omega}(x \boxplus_q y)$ durch die Formel (49) definiert wird.

Satz 3.9. *Die Menge* $(\mathcal{M}_{x,q,\omega}, \cdot, \cdot_{q,\omega}, \mathrm{I}_n)$ *mit Multiplikationen gegeben durch (76) und (78), und Inverse* $\overline{\Phi}^{(-M)}_{n,q,\omega}(-x)$ *ist eine* q,ω*-Liesche Gruppe. Das Einheitselement ist die Einheitsmatrix* I_n *und das assoziative Gesetz gilt wie für Gruppen.*

Wir geben eine vereinfachte Version des entsprechenden Nachweises.

Beweis. $\mathcal{M}_{x,q,\omega}$ ist geschlossen unter den beiden Operationen durch (76) und (78). Durch (78) haben wir

$$
\overline{\Phi}^{(M)}_{n,q,\omega}(x) \cdot_q \overline{\Phi}^{(-M)}_{n,q,\omega}(-x) = \overline{\Phi}^{(0)}_{n,q,\omega}(\theta) = \mathrm{I}_n, \tag{79}
$$

das die Existenz eines inversen Elements und einer Einheit beweist.

Zur Vereinfachung der Notation wird der letzte Teil in einem Spezialfall angegeben, das leicht zu verallgemeinern ist. Das assoziative Gesetz lautet:

$$(80)\quad \left(\overline{\Phi}^{(M)}_{n,q,\omega}(x)\cdot\overline{\Phi}^{(K)}_{n,q,\omega}(y)\right)\cdot_q\overline{\Phi}^{(J)}_{n,q,\omega}(z)=\overline{\Phi}^{(M)}_{n,q,\omega}(x)\cdot\left(\overline{\Phi}^{(K)}_{n,q,\omega}(y)\cdot_q\overline{\Phi}^{(J)}_{n,q,\omega}(z)\right),$$

das äquivalent zu

$$(81)\quad \overline{\Phi}^{(M+K+J)}_{n,q,\omega}((x\oplus_{q,\omega}y)\boxplus_{q,\omega}z)=\overline{\Phi}^{(M+K+J)}_{n,q,\omega}(x\oplus_{q,\omega}(y\boxplus_{q,\omega}z))$$

ist. Die Formel (81) folgt jedoch aus der Assoziativität der beiden q,ω-Additionen. □

Sei

$$(82)\quad \left(\overline{\Phi}^{(M)}_{n,q,\omega}(x)\right)^k\equiv\overline{\Phi}^{(M)}_{n,q,\omega}(x)\cdot\overline{\Phi}^{(M)}_{n,q,\omega}(x)\cdot\cdots\cdot\overline{\Phi}^{(M)}_{n,q,\omega}(x).$$

Dabei steht auf der rechten Seite das Produkt von k gleichen Matrizen $\overline{\Phi}^{(M)}_{n,q}(x)$.

Die Formel (63) kann zu

$$(83)\quad \left(\overline{\Phi}^{(M)}_{n,q,\omega}(x)\right)^k=\overline{\Phi}^{(kM)}_{n,q,\omega}(\overline{k}_{q,\omega}x)$$

verallgemeinert werden.

3.2. **q,ω-Bernoulli und q,ω-Eulersche Polynome.** Wir betrachten auch die besonderen Fälle q,ω-Bernoulli- und q,ω-Eulersche Polynome.

Definition 26. Es gibt zwei q,ω-Bernoulli-Polynome $\mathrm{B}_{\mathrm{NWA},\nu,q,\omega}(x)$, NWA q,ω-Bernoulli-Polynome, und $\mathrm{B}_{\mathrm{JHC},\nu,q,\omega}(x)$, JHC q,ω-Bernoulli-Polynome. Sie sind definiert durch die beiden erzeugenden Funktionen

$$(84)\quad \frac{t}{(\mathrm{E}_{q,\omega}(t)-1)}\mathrm{E}_{q,\omega t}(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu\mathrm{B}_{\mathrm{NWA},\nu,q}(x)}{\{\nu\}_q!},\ |t|<2\pi.$$

und

$$(85)\quad \frac{t}{(\mathrm{E}_{\frac{1}{q},\omega}(t)-1)}\mathrm{E}_{q,\omega t}(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu\mathrm{B}_{\mathrm{JHC},\nu,q}(x)}{\{\nu\}_q!},\ |t|<2\pi.$$

Definition 27. Die Ward q,ω-Bernoullischen Zahlen sind gegeben durch

$$(86)\quad \mathrm{B}_{\mathrm{NWA},n,q,\omega}\equiv\mathrm{B}_{\mathrm{NWA},n,q,\omega}(0).$$

Die Jackson q,ω-Bernoullischen Zahlen sind gegeben durch

$$\mathrm{B}_{\mathrm{JHC},n,q,\omega} \equiv \mathrm{B}_{\mathrm{JHC},n,q,\omega}(0). \tag{87}$$

Um Platz zu sparen, verwenden wir die folgende Abkürzung in den Gleichungen (89) - (93), (96), (97), (100), (104), (105), (108)-(112), (115)-(116), (119)-(120). Für den JHC-Fall ändern wir gegebenenfalls $\oplus_{q,\omega}$ zu $\boxplus_{q,\omega}$.

$$\mathrm{NWA} = \mathrm{NWA} \vee \mathrm{JHC}. \tag{88}$$

Wir werden die folgenden Vektorformen für die q,ω-Bernoulli-Polynome verwenden, die q,ω-Analoga von [1, S. 239] entsprechen.

$$b_{\mathrm{NWA},n,q,\omega}(x) \equiv (\mathrm{B}_{\mathrm{NWA},0,q,\omega}(x), \mathrm{B}_{\mathrm{NWA},1,q,\omega}(x), \ldots, \mathrm{B}_{\mathrm{NWA},n-1,q,\omega}(x))^T. \tag{89}$$

Die entsprechenden Vektorformen für Zahlen sind

$$b_{\mathrm{NWA},n,q,\omega} \equiv (\mathrm{B}_{\mathrm{NWA},0,q,\omega}, \mathrm{B}_{\mathrm{NWA},1,q,\omega}, \ldots, \mathrm{B}_{\mathrm{NWA},n-1,q,\omega})^T. \tag{90}$$

Wir stellen die NWA und JHC verschobenen, q,ω-Bernoulli-Matrizen vor.

Definition 28. Ein ω-Analogon von [9, (54)].

$$\begin{aligned} &\mathcal{B}_{\mathrm{NWA},n,q,\omega}(x) \\ \equiv&(b_{\mathrm{NWA},q,\omega}(x) \;\; \mathrm{E}(\oplus_{q,\omega})b_{\mathrm{NWA},q,\omega}(x) \;\cdots\; \mathrm{E}(\oplus_{q,\omega})^{\overline{n-1}_{q,\omega}} b_{\mathrm{NWA},q,\omega}(x)), \end{aligned} \tag{91}$$

wobei $\mathrm{E}(\oplus_{q,\omega})^{\overline{k}_{q,\omega}}((x)^n_{q,\omega}) \equiv (x \oplus_{q,\omega} \overline{k}_{q,\omega})^n$, $0 \le k \le n-1$.

Wir benötigen zwei ähnliche Matrizen basierend auf den $\mathrm{B}_{\mathrm{NWA}}$ und $\mathrm{B}_{\mathrm{JHC}}$-Polynomen und Zahlen.

Definition 29. Zwei ω-Analoga von [9, (58),(67)] und zwei q,ω-Analoga von [4, S. 193]. Die NWA und JHC q,ω-Bernoulli-Matrizen sind definiert durch

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q \mathrm{B}_{\mathrm{NWA},i-j,q,\omega}(x), 0 \le i,j \le n-1. \tag{92}$$

Definition 30. Ein ω-Analogon von [10, 84]. Die NWA und JHC q,ω-Bernoulli-Zahlenmatrizen sind definiert durch

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(i,j) \equiv \binom{i}{j}_q \mathrm{B}_{\mathrm{NWA},i-j,q,\omega}, 0 \le i,j \le n-1. \tag{93}$$

Definition 31. Ein ω-Analogon von [10, 85]. Die Matrix $\mathcal{D}_{\mathrm{NWA},n,q,\omega}$ hat Matrixelemente

$$d_{\mathrm{NWA},i,j} \equiv \begin{cases} \frac{1}{\{i-j+1\}_q} \binom{i}{j}_q (1)_{q,\omega}^{i-j+1} \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{94}$$

Definition 32. Ein ω-Analogon von [10, 86]. Die Matrix $\mathcal{D}_{\mathrm{JHC},n,q,\omega}$ hat Matrixelemente

$$d_{\mathrm{JHC},i,j} \equiv \begin{cases} \frac{1}{\{i-j+1\}_q} \binom{i}{j}_q [1]_{q,\omega}^{i-j+1} \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{95}$$

Satz 3.10. *Ein ω-Analogon von* [10, 87]. *Die Inversen der q,ω-Bernoulli-Zahlenmatrizen sind gegeben durch*

$$\left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}\right)^{-1} = \mathcal{D}_{\mathrm{NWA},n,q,\omega}, \ \left(\overline{\mathrm{B}}_{\mathrm{JHC},n,q,\omega}\right)^{-1} = \mathcal{D}_{\mathrm{JHC},n,q,\omega}. \tag{96}$$

Dies impliziert, dass

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}^{-k} = \mathcal{D}_{\mathrm{NWA},n,q,\omega}^{k}. \tag{97}$$

Beweis. Wir betrachten nur den NWA-Fall, für JHC, ändere man zu $[1]_{q,\omega}^{i-j+1}$. Wir zeigen, dass $\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}\mathcal{D}_{\mathrm{NWA},n,q,\omega}$ gleich der Einheitsmatrix ist. Wir wissen, dass

$$\sum_{k=0}^{n} \frac{1}{\{k+1\}_q} \binom{n}{k}_q \mathrm{B}_{\mathrm{NWA},n-k,q,\omega} (1)_{q,\omega}^{k+1} = \delta_{n,0}. \tag{98}$$

Dann haben wir

(99)

$$\begin{aligned} &\sum_{k=j}^{i} \frac{1}{\{k+1-j\}_q} \binom{i}{k}_q \mathrm{B}_{\mathrm{NWA},i-k,q,\omega} \binom{k}{j}_q (1)_{q,\omega}^{k+1-j} \\ &= \binom{i}{j}_q \sum_{k=j}^{i} \frac{1}{\{k+1-j\}_q} \binom{i-j}{k-j}_q \mathrm{B}_{\mathrm{NWA},i-k,q,\omega} (1)_{q,\omega}^{k+1-j} \\ &= \binom{i}{j}_q \sum_{k=0}^{i-j} \frac{1}{\{k+1\}_q} \binom{i-j}{k}_q \mathrm{B}_{\mathrm{NWA},i-j-k,q,\omega} (1)_{q,\omega}^{k+1} \overset{\text{durch}(98)}{=} \binom{i}{j}_q \delta_{i-j,0}. \end{aligned}$$

□

In [9] haben wir die folgenden Formeln betrachtet.

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(x \oplus_q y) = \mathrm{P}_{n,q}(x)\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(y). \tag{100}$$

Diese Formeln können verallgemeinert werden zu

Satz 3.11. *Ein ω-Analogon von* [10, 92].

$$\overline{\Phi}_{n,q,\omega}(x \oplus_{q,\omega} y) = \mathrm{P}_{n,q,\omega}(x)\overline{\Phi}_{n,q,\omega}(y). \tag{101}$$

Insbesondere haben wir

$$\overline{\Phi}_{n,q,\omega}(x) = \mathrm{P}_{n,q,\omega}(x)\overline{\Phi}_{n,q,\omega}. \tag{102}$$

Beweis.

$$\begin{aligned}
\mathrm{RS} &= \sum_{k=j}^{i} \binom{i}{k}_q (x)_{q,\omega}^{i-k} \binom{k}{j}_q \Phi_{k-j,q,\omega}(y) \\
&= \binom{i}{j}_q \sum_{k=j}^{i} \binom{i-j}{k-j}_q (x)_{q,\omega}^{i-k} \Phi_{k-j,q,\omega}(y) \\
&= \binom{i}{j}_q \sum_{k=0}^{i-j} \binom{i-j}{k}_q (x)_{q,\omega}^{i-j-k} (x) \Phi_{k,q,\omega}(y) \\
&\overset{\text{durch}(43),(47)}{=} \binom{i}{j}_q \Phi_{i-j,q,\omega}(x \oplus_{q,\omega} y) = \mathrm{LS}.
\end{aligned} \tag{103}$$

□

Satz 3.12. *Ein ω-Analogon von* [10, 95]. *Die Inversen der q,ω-Bernoulli-Polynommatrizen sind gegeben durch*

$$\left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(x)\right)^{-1} = \left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}\right)^{-1} \mathrm{P}_{n,q,\omega}(x)^{-1} = \mathcal{D}_{\mathrm{NWA},n,q,\omega}\mathrm{P}_{n,q,\omega}(x)^{-1}. \tag{104}$$

Wenn die Ordnung erhöht wird, für $y = 0$ in (76), multiplizieren wir die q,ω-Übertragung-Matrix mit $\overline{\Phi}^{(M)}_{n,q,\omega}(x)$. Wenn die Ordnung konstant ist, in (102), multiplizieren wir die q,ω-Übertragung-Matrix mit der q,ω-Pascal-Matrix.

Satz 3.13. *Die Funktionen der q,ω-Bernoulli Polynommatrizen* $(\mathcal{B}_{\mathrm{NWA},q,\omega},\cdot,\cdot_q,\mathrm{I}_n)$ *und* $(\mathcal{B}_{\mathrm{JHC},q,\omega},\cdot,\cdot_q,\mathrm{I}_n)$ *mit Elementen*

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(x) \tag{105}$$

sind q,ω-Liesche Untergruppen von $\mathcal{M}_{x,q}$.

Beweis. Die Mengen $\mathcal{B}$ sind geschlossen unter den beiden Operationen durch (76) und (78). Die Existenz von Inversen folgt wie für $\mathcal{M}_{x,q}$.

□

Wir wenden uns nun den q,ω-Euler-Polynomen zu.

Definition 33. Es gibt zwei Arten von q,ω-Euler-Polynomen, $\mathrm{F}_{\mathrm{NWA},\nu,q,\omega}(x)$, NWA q,ω-Euler-Polynomen, und $\mathrm{F}_{\mathrm{JHC},\nu,q,\omega}(x)$, JHC q,ω-Euler-Polynomen. Sie sind definiert durch die folgenden zwei erzeugenden Funktionen:

$$\frac{2\mathrm{E}_{q,\omega t}(xt)}{\mathrm{E}_{q,\omega}(t)+1}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\mathrm{F}_{\mathrm{NWA},\nu,q,\omega}(x),\ |t|<\pi, \tag{106}$$

und

$$\frac{2\mathrm{E}_{q,\omega t}(xt)}{\mathrm{E}_{\frac{1}{q},\omega}(t)+1}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\mathrm{F}_{\mathrm{JHC},\nu,q,\omega}(x),\ |t|<\pi. \tag{107}$$

Definition 34. Wir werden die folgenden Vektorformen für diese Polynome benutzen.

$$f_{\mathrm{NWA},n,q,\omega}(x)\equiv(\mathrm{F}_{\mathrm{NWA},0,q,\omega}(x),\mathrm{F}_{\mathrm{NWA},1,q,\omega}(x),\ldots,\mathrm{F}_{\mathrm{NWA},n-1,q,\omega}(x))^T. \tag{108}$$

Die entsprechenden q,ω-Euler-Zahlvektoren sind

$$f_{\mathrm{NWA},n,q,\omega}\equiv(\mathrm{F}_{\mathrm{NWA},0,q,\omega},\mathrm{F}_{\mathrm{NWA},1,q,\omega},\ldots,\mathrm{F}_{\mathrm{NWA},n-1,q,\omega})^T. \tag{109}$$

Wir stellen die NWA und JHC verschobenen, q,ω-Eulerschen Matrizen vor.

Definition 35. Ein ω-Analogon von [10, 103].

$$\begin{aligned}&\mathcal{F}_{\mathrm{NWA},n,q,\omega}(x)\\ &\equiv(f_{\mathrm{NWA},q,\omega}(x)\ \ \mathrm{E}(\oplus_{q,\omega})f_{\mathrm{NWA},q,\omega}(x)\ \cdots\ \mathrm{E}(\oplus_{q,\omega})^{\overline{n-1}_{q,\omega}}f_{\mathrm{NWA},q,\omega}(x)).\end{aligned} \tag{110}$$

Wir benötigen zwei ähnliche Matrizen, basierend auf den $\mathrm{F}_{\mathrm{NWA}}$-Polynomen.

Definition 36. Die beiden q,ω-Eulerschen Polynommatrizen sind definiert durch

$$\overline{F}_{\mathrm{NWA},n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q f_{\mathrm{NWA},i-j,q,\omega}(x). \tag{111}$$

Definition 37. Ein ω-Analogon von [10, 105]. Die NWA und JHC q,ω-Eulerschen Matrizen sind definiert durch

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}(i,j) \equiv \binom{i}{j}_q \mathrm{F}_{\mathrm{NWA},i-j,q,\omega}, 0 \leq i,j \leq n-1. \tag{112}$$

Definition 38. Ein ω-Analogon von [10, 106]. Die Matrix $\mathcal{C}_{\mathrm{NWA},n,q,\omega}$ hat Matrixelemente

$$c_{\mathrm{NWA},i,j} \equiv \begin{cases} \frac{(1)_{q,\omega}^{i-j+1}+\delta_{i-j,0}}{2}\binom{i}{j}_q \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{113}$$

Definition 39. Ein ω-Analogon von [10, 107]. Die Matrix $\mathcal{C}_{\mathrm{JHC},n,q,\omega}$ hat Matrixelemente

$$c_{\mathrm{JHC},i,j} \equiv \begin{cases} \frac{[1]_{q,\omega}^{i-j+1}+\delta_{i-j,0}}{2}\binom{i}{j}_q \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{114}$$

Satz 3.14. *Ein ω-Analogon von* [10, 108]. *Die Inversen der q,ω-Euler-Zahlenmatrizen sind gegeben durch*

$$\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}\right)^{-1} = \mathcal{C}_{\mathrm{NWA},n,q,\omega}. \tag{115}$$

Dies impliziert, dass

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}^{-k} = \mathcal{C}_{\mathrm{NWA},n,q,\omega}^{k}. \tag{116}$$

Beweis. Wir betrachten nur den NWA-Fall, für JHC, ändere man zu $[1]_{q,\omega}^{i-j+1}$. Wir zeigen, dass $\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}\mathcal{C}_{\mathrm{NWA},n,q,\omega}$ gleich der Einheitsmatrix ist. Wir wissen, dass

$$\sum_{k=0}^{n}(1)_{q,\omega}^{k}\binom{n}{k}_q \mathrm{F}_{\mathrm{NWA},n-k,q} + \mathrm{F}_{\mathrm{NWA},n,q} = 2\delta_{n,0}. \tag{117}$$

Man führe eine Funktion $G(k)$ ein. Dann haben wir

$$\begin{aligned}
&\sum_{k=j}^{i} \binom{i}{k}_q \mathrm{F}_{\mathrm{NWA},i-k,q,\omega} G(k-j) \binom{k}{j}_q \\
&= \binom{i}{j}_q \sum_{k=j}^{i} \binom{i-j}{k-j}_q \mathrm{F}_{\mathrm{NWA},i-k,q,\omega} G(k-j) \qquad (118) \\
&= \binom{i}{j}_q \sum_{k=0}^{i-j} \binom{i-j}{k}_q \mathrm{F}_{\mathrm{NWA},i-j-k,q,\omega} G(k) \overset{\text{durch}(117)}{=} \binom{i}{j}_q \delta_{i-j,0}.
\end{aligned}$$

Es ist jetzt offensichtlich, dass $G(k) = \frac{1}{2}\left[(1)_{q,\omega}^k + \delta_{k,0}\right]$ diese Gleichung für NWA löst, und dass der JHC-Fall ähnlich gelöst wird. □

Die letzten beiden Sätze werden auf ähnliche Weise belegt.

Satz 3.15. *Die Inversen der q,ω-Eulerschen Polynommatrizen sind gegeben durch*

$$\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}(x)\right)^{-1} = \left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}\right)^{-1} \mathrm{P}_{n,q,\omega}(x)^{-1} = \mathrm{C}_{\mathrm{NWA},n,q} \mathrm{P}_{n,q,\omega}(x)^{-1}. \qquad (119)$$

Satz 3.16. *Ein ω-Analogon von* [10, 114]. *Die q,ω-Eulerschen Polynommatrizen $(\mathcal{F}_{\mathrm{NWA},q}, \cdot, \cdot_q, \mathrm{I}_n)$ und $(\mathcal{F}_{\mathrm{JHC},q}, \cdot, \cdot_q, \mathrm{I}_n)$ mit Elementen*

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q}(x) \qquad (120)$$

sind q,ω-Lieschen Untergruppen von $\mathcal{M}_{x,q}$.

4. Schlussfolgerung

Wir haben Formeln aus Arbeiten von Arponen [4], Aceto et al. [2], Ernst [8], [9], [10], vereinigt und q,ω-deformiert, um eine erste Synthese von q,ω-Appell-Polynommatrizen vorzustellen. Einige Formeln für q-Pascal -Matrizen sowie Formeln für q-Bernoulli- und q-Eulerschen Matrizen werden verallgemeinert. Wir haben die ersten konkreten Beispiele von q,ω-Lieschen Untergruppen angegeben. Wir glauben, dass es keine weitere Analoga gibt, die q,ω-Analoga sind trotzdem sehr interessant.

LITERATUR

[1] L. Aceto, D. Trigiante, The matrices of Pascal and other greats. *Amer. Math. Monthly* 108 no. 3, 232–245 (2001)

[2] L. Aceto, H.R. Malonek and G. Tomaz, A unified matrix approach to the representation of Appell polynomials, *Integral Transforms Spec. Funct.* 26 no. 6 , 426–441 (2015)

[3] M. H. Annaby, A. E. Hamza, K. A. Aldwoah, Hahn difference operator and associated Jackson-Nørlund integrals. *J. Optim. Theory Appl. 154, No. 1, 133-153 (2012)*

[4] T. Arponen, A matrix approach to polynomials, Linear Algebra Appl. 359, 181–196 (2003)

[5] T.Ernst, *A comprehensive treatment of q-calculus,* Birkhäuser (2012)

[6] T.Ernst, An umbral approach to find q-analogues of matrix formulas. *Linear Algebra Appl.* **439**, 1167–1182 (2013)

[7] T.Ernst, Faktorisierungen von q-Pascalmatrizen (Factorizations of q-Pascal matrices). *Algebras Groups Geom. 31 no. 4, ,387-405 (2014)*

[8] T.Ernst, On the q-exponential of matrix q-Lie algebras. Spec. Matrices 5, 36-50 (2017)

[9] T.Ernst, On several q-special matrices, including the q-Bernoulli and q-Euler matrices. *Linear Algebra Appl. 542 , 422-440 (2018)*

[10] T.Ernst, On the q-Lie group of q-Appell polynomial matrices and related factorizations Spec. Matrices 6, 93-109 (2018)

[11] W.Hahn, Über Polynome, die gleichzeitig zwei verschiedenen Orthogonalsystemen angehören. (German) *Math. Nachr. 2, 263-278 (1949)*

[12] W. P. Johnson, q-extension of identities of Abel-Rothe type. Discrete Math. 159, No. 1-3, 161-177 (1996)

[13] S. Varma, B. Yasar, M. Özarslan, Hahn-Appell polynomials and their d-orthogonality *Revista de la Real Academia de Ciencias Ex* (2018)

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DOI 10.29083/AGG.35.03.2018

(ε, δ)-FREUDENTHAL KANTOR TRIPLE SYSTEMS, δ-STRUCTURABLE ALGEBRAS AND LIE SUPERALGEBRAS

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Received November 7, 2019

Abstract

In this paper we discuss (ε, δ)-Freudenthal Kantor triple systems with certain structure on the subspace L_{-2} of the corresponding standard embedding five graded Lie (super)algebra $L(\varepsilon, \delta) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, $[L_i, L_j] \subseteq L_{i+j}$. We recall Lie and Jordan structures associated with (ε, δ)-Freudenthal Kantor triple systems ([26], [27]) and the give results for unitary and pseudo-unitary (ε, δ)-Freudenthal Kantor triple systems. Further, we give the notion of δ-structurable algebras and connect them to $(-1, \delta)$-Freudenthal Kantor triple systems and the corresponding Lie (super)algebra construction.

AMS 2000 Mathematics subject classification: Primary 17A40; 17B60

Keywords: Lie superalgebras; triple systems.

[1]Research partially supported by Grant-in-Aid for Scientific Research (No. 19540042 (C),(2)), Japan Society for the Promotion of Science.
[2]Research supported by U.S. Department of Energy Grant No. DE-FG02-91ER40685.

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1 Introduction

Nonassociative algebras are rich in algebraic structures and provide an important common ground for various branches of mathematics. Specially, Jordan and Lie (super)algebras plays an important role in many mathematical and physical subjects ([9]-[12], [14], [22], [24], [39], [43]). We note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([18], [40]) by using the standard embedding method ([20], [33], [34], [41], [45]). In particular, the generalized Jordan triple system of second order (that is, the $(-1,1)$-Freudenthal Kantor triple system) is a useful concept ([12]-[19], [29]-[32], [44]) for the constructions of simple Lie algebras, while the $(-1,-1)$-Freudenthal Kantor triple system plays the same role ([5], [20], [21], [23]) for the construction of Lie superalgebras. The purpose of this paper is to recall Lie and Jordan structures associated with (ε,δ)-Freudenthal Kantor triple systems ([26], [27]) and the give results for unitary and pseudo-unitary (ε,δ)-Freudenthal Kantor triple systems. Further, we introduce the notion of δ-structurable algebras and connect them to $(-1,\delta)$-Freudenthal Kantor triple systems and the corresponding Lie (super)algebra construction.

2 Definitions, preliminaries, Lie and Jordan structures

2.1 (ϵ,δ)-Freudenthal Kantor triple systems, δ-Lie triple systems and Lie (super)algebras

We are concerned in this paper with triple systems which have finite dimension over a field Φ of characteristic $\neq 2$ or 3.

We recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order).

A vector space V over a field Φ endowed with a trilinear operation $V\times V\times V\to V$, $(x,y,z)\longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{1}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0, \tag{2}$$

where $L(a,b)c := (abc)$ and $K(a,b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1) and the identity

$$(abc) = (cba). \tag{3}$$

We can generalize the concept of GJTS of 2nd order as follows (see [12], [13], [16], [20], [45] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \tag{4}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) + \varepsilon K(a,b)L(x,y) = 0, \tag{5}$$

where

$$L(a,b)c := (abc), \quad K(a,b)c := (acb) - \delta(bca), \tag{6}$$

is called an (ε,δ)-*Freudenthal Kantor triple system* (for short (ε,δ)-FKTS).

Remark. We note that $K(b,a) = -\delta K(a,b)$.

From now on we will mainly deal with this type of triple system.

An (ε,δ)-FKTS is said to be *balanced* if it fulfills $dim_{\Phi}\{K(a,b)\}_{span} = 1$.

An (ε,δ)-FKTS U is called *unitary* if the identity map Id is contained in $\kappa := K(U,U)$ i.e., if there exist $a_i, b_i \in U$, such that $\Sigma_i K(a_i,b_i) = Id$.

Remark. A balanced triple system is unitary, since $Id \in \kappa = K(U,U)$.

Triple products are generally denoted (xyz), $\{xyz\}$, $[xyz]$ and $< xyz >$.

Remark. The concept of GJTS of 2nd order coincides with that of $(-1,1)$-FKTS. Thus we can construct the simple Lie algebras by means of the standard embedding method ([5], [12]-[16], [20], [21], [23], [31], [45]).

Remark. For an (ε,δ)-FKTS U we denote

$$S(a,b) := L(a,b) + \varepsilon L(b,a), \quad A(a,b) := L(a,b) - \varepsilon L(b,a), \tag{7}$$

where $L(a,b)$ is defined by (6).

Remark. We note that $S(a,b) = \varepsilon S(b,a)$. Then $S(a,b)$ (respectively $A(a,b)$) is a derivation (respectively anti-derivation) of U ([26]), that is

$$[S(a,b), L(c,d)] = L(S(a,b)c,d) + L(c,S(a,b)d),$$
$$[A(a,b), L(c,d)] = L(A(a,b)c,d) - L(c,A(a,b)d).$$

For $\delta = \pm 1$, a triple system $(a,b,c) \mapsto [abc], a,b,c \in V$ is called a *δ-Lie triple system* (for short δ-LTS) if the following three identities are fulfilled

$$\begin{array}{c} [abc] = -\delta[bac], \\ [abc] + [bca] + [cab] = 0, \\ [ab[xyz]] = [[abx]yz] + [x[aby]z] + [xy[abz]], \end{array} \tag{8}$$

$a,b,x,y,z \in V$. An 1-LTS is a *LTS* while a -1-LTS is an *anti-LTS* ([13]).

Proposition 2.1 *([13],[20]) Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J < xyz >=< JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta),[xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] :=< xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > . \tag{9}$$

Corollary 2.1 *([13],[20]) Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. Then the vector space $T(\varepsilon,\delta) = U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product defined by*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}.$$

Thus we obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$), $L(\varepsilon,\delta) = D(T(\varepsilon,\delta),T(\varepsilon,\delta)) \oplus T(\varepsilon,\delta)$, associated to $T(\varepsilon,\delta)$ where $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$, i.e.

$$D(T(\varepsilon,\delta),T(\varepsilon,\delta)) := \left\{\begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix}\right\}_{span},$$

$$T(\varepsilon,\delta) := \left\{\begin{pmatrix} x \\ y \end{pmatrix}\middle| x,y \in U(\varepsilon,\delta)\right\}_{span}.$$

Remark. We note that $L(\varepsilon,\delta) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is the five graded Lie (super)algebra, such that $L_{-1} \oplus L_1 = T(\varepsilon,\delta)$ and $D(T(\varepsilon,\delta),T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_2$ with $[L_i,L_j] \subseteq L_{i+j}$.

Proposition 2.1 is generalized as follows.

Theorem 2.1 *([27]) Let $(U,(xyz))$ be a (ε,δ)-FKTS and P be an endomorphism of U such that $P^2=\mu Id$ and $P(xyz)=\nu(Px)(Py)(Pz)$, $\mu,\nu=\pm 1$. Define the new triple product by*

$$[x,y,z]:=x(Py)z+\varepsilon^*y(Px)z-\varepsilon^*x(Pz)y+\varepsilon^*\delta y(Pz)x,$$

where $\varepsilon^*=\mu\nu\varepsilon$. If $\mu\nu\varepsilon\delta=-1$, then $(U(\varepsilon,\delta),[xyz])$ is a LTS (if $\delta=1$) or an anti-LTS (if $\delta=-1$).

2.2 JTSs associated with (ε,δ)-FKTSs

In this section, we consider the properties of $\{K(x,y)\}_{span}$ defined by (6).

Proposition 2.2 *([26]) Let U be an (ε,δ)-FKTS. Then we have*

$$K(u,v)K(x,y) = L(v,K(x,y)u)-\delta L(u,K(x,y)v) \tag{10}$$
$$= \varepsilon\delta L(K(u,v)y,x)-\varepsilon L(K(u,v)x,y) \tag{11}$$

for any $u,v,x,y\in U$, where $L(x,y),K(x,y)$ are defined by (6).

Corollary 2.2 *([26]) The $K(x,y)$ satisfies a Lie relation type, that is*
$[K(x,y),[K(u,v),K(a,b)]]=$
$\delta K([K(u,v),K(a,b)]y,x)-K([K(u,v),K(a,b)]x,y)$
and the following identities are valid for $S(a,b)$ defined by (7):

$$[K(a,b),K(x,y)]=\delta S(x,K(a,b)y)-\varepsilon S(K(a,b)x,y), \tag{12}$$
$$[S(x,y),K(a,b)]=K(K(a,b)y,x)+\varepsilon K(K(a,b)x,y), \tag{13}$$
$$=K(S(x,y)a,b)+K(a,S(x,y)b), \tag{14}$$
$$[S(a,b),S(x,y)]=S(S(a,b)x,y)+S(x,S(a,b)y). \tag{15}$$

Remark. The Corollary 2.2 shows that the $K(a,b)$ defined by (6) has a structure of a LTS with respect to the product

$$[[K(a,b),K(c,d)],K(e,f)]=[K(a,b),K(c,d),K(e,f)].$$

Proposition 2.3 *([26]) Let U be an (ε,δ)-FKTS. Then the $K(x,y)$ satisfies a JTS relation type, that is*

$$\begin{aligned} &K(a,b)K(x,y)K(u,v)+K(u,v)K(x,y)K(a,b)\\ &=K(K(a,b)K(x,y)u,v)-\delta K(K(a,b)K(x,y)v,u), \qquad (16)\\ &=\varepsilon\delta K(K(a,b)x,K(u,v)y)-\varepsilon K(K(a,b)y,K(u,v)x). \qquad (17)\end{aligned}$$

Let $\kappa=\{K(x,y)|x,y\in U\}_{span}$ and define a triple product in κ by

$$\{K_1,K_2,K_3\}:=K_1K_2K_3+K_3K_2K_1,\ (K_j\in\kappa). \qquad (18)$$

Remark. From Proposition 2.3 it follows then that

$$\{K(a,b),K(x,y),K(c,d)\}=K(K(a,b)K(x,y)c,d)-\delta K(K(a,b)K(x,y)d,c).$$

Proposition 2.4 *([26]) The triple product $\{K_1,K_2,K_3\}$ in (18) is a JTS.*

Proposition 2.5 *([26]) For the triple product $\{,,,\}$ defined by (18) let $\sigma(x,y)\in End\ \kappa$ and $\theta(x,y)\in End\ \kappa, x,y\in U$, be defined by*

$$\sigma(x,y)K(a,b):=K(K(a,b)x,y)-\varepsilon\delta K(x,K(a,b)y), \qquad (19)$$

$$\theta(x,y)K(a,b):=K(K(a,b)x,y)+\varepsilon\delta K(x,K(a,b)y). \qquad (20)$$

Then $\sigma(x,y)$ is a derivation and $\theta(x,y)$ is an anti-derivation of the JTS κ.

If we assume the unitary property, we have the following results:

Proposition 2.6 *Let U be an unitary (ε,δ)-FKTS. Then we have*

$$(i)\ \varepsilon=\delta\ (or\ \varepsilon\delta=1),$$

$$(ii)\ K(x,y)=L(y,x)-\varepsilon L(x,y)=-\varepsilon A(x,y), \qquad (21)$$

$$\begin{aligned}&(iii)\ K(x,y)K(u,v)+K(u,v)K(x,y)=K(K(u,v)x,y)+\\ &K(x,K(u,v)y)=K(u,K(x,y)v)+K(K(x,y)u,v), \qquad (22)\end{aligned}$$

$$(iv)\ K(x,y)\text{ is an anti}-\text{derivation of }U.$$

Proposition 2.7 *([26]) Let U be an unitary (ε, δ)-FKTS. Then the commutative product in $K(U,U)$ defined by*

$$K(u,v) * K(x,y) = K(u,v)K(x,y) + K(x,y)K(u,v) =$$

$$K(x, K(u,v)y) + K(K(u,v)x, y) \tag{23}$$

defines a Jordan algebra κ^*. Moreover, $\sigma(x,y)$ is a derivation of κ^*.

Remark. We note that κ's property is the same to the property of $L_{-2} = \{\left(\begin{smallmatrix} 0 & K(x,y) \\ 0 & 0 \end{smallmatrix}\right) | x, y \in U\}_{span}$, thus the investigation of the (ε, δ)-FKTS U means the study of the standard embedding Lie (super)algebra.

2.3 Pseudo unitarity

Let U be an (ε, δ)-FKTS. For $\varepsilon\delta = -1$, we can not have unitarity, so we note ([27]) that we can relax the condition as follows. Let $P \in \mathrm{End}U$ with

$$P^2 = \mu Id, \quad \mu = \pm 1, \tag{24}$$

$$P(xyz) = \nu(Px)(Py)(Pz), \quad \nu = \pm 1. \tag{25}$$

We note then that the triple system $(U,\ (xyz)^*)$ with the new triple product $(xyz)^* := x(Py)z$ is an (ε^*, δ)-FKTS with $\varepsilon^* = \mu\nu\varepsilon$ ([27]). We denote $(U,\ (xyz)^*)$ by U^*. Therefore, if $\varepsilon = -\delta$ and if we can choose $\mu\nu = -1$, then U^* is a (δ, δ)-FKTS, so that it can be unitary ([27]).

Let U be an (ε, δ)-FKTS and let $P \in \mathrm{End}U$. We call U to be *pseudo unitary* if there are $\nu,\ \mu \in \{\pm 1\}$ and $P \in \kappa = K(U,U)$ (i.e., if there exist $a_i, b_i \in U$, such that $\Sigma_i K(a_i, b_i) = P$) and P satisfies (24).

We show now that (25) is a consequence of $P \in \kappa$ and (24) by proving the following lemma end proposition.

Lemma 2.1 *Let U be an (ε, δ)-FKTS and let $P \in \mathrm{End}U$ such that $P \in \kappa = K(U,U)$. Then we have*

$$K(Px, y) = L(y,x)P - \varepsilon PL(x,y). \tag{26}$$

Proof. Indeed, we have $K(K(a,b)x, y) = L(y,x)K(a,b) - \varepsilon K(a,b)L(x,y)$, by (5). Put now $a = a_i, b = b_i$ in the last identity and summing over i we obtain (26).

Proposition 2.8 *Let U be an (ε,δ)-FKTS and let $P \in \mathrm{End}U$ such that $P \in \kappa = K(U,U)$ and (24) is fulfilled. Then we have*

$$P(xyz) = \mu\varepsilon\delta(Px)(Py)(Pz) = \mu\delta yx(Pz) + \mu yz(Px) - \mu\delta(Px)zy. \quad (27)$$

Furthermore, if $\mu\varepsilon\delta = 1$, then P is an automorphism of U.

Proof. By (26), we have $(Px)zy - \delta yz(Px) = yx(Pz) - \varepsilon P(xyz)$, or

$$P(xyz) = -\varepsilon(Px)zy + \varepsilon yx(Pz) + \varepsilon\delta yz(Px). \quad (28)$$

Applying P again to the identity (28) we have, by (24),

$$\mu xyz = P^2(xyz) = -\varepsilon P((Px)zy) + \varepsilon P(yx(Pz)) + \varepsilon\delta P(yz(Px)). \quad (29)$$

Changing $x \to Px$ and $y \leftrightarrow z$ in (28) and multiplying with $-\varepsilon$ follows $-\varepsilon P((Px)zy) = -\varepsilon[-\varepsilon(P^2x)yz + \varepsilon z(Px)(Py) + \varepsilon\delta zy(P^2x)]$ hence, by (24),

$$-\varepsilon P((Px)zy) = \mu xyz - z(Px)(Py) - \mu\delta zyx. \quad (30)$$

Changing $x \leftrightarrow y$ and $z \to Pz$ in (28) and multiplying with ε follows $\varepsilon P(yx(Pz)) = \varepsilon[-\varepsilon(Py)(Pz)x + \varepsilon xy(P^2z) + \varepsilon\delta x(Pz)(Py)]$ hence, by (24),

$$\varepsilon P(yx(Pz)) = -(Py)(Pz)x + \mu xyz + \delta x(Pz)(Py). \quad (31)$$

Finally, changing $x \to y \to z \to Px$ in (28) and multiplying with $\varepsilon\delta$ follows $\varepsilon\delta P(yz(Px)) = \varepsilon\delta[-\varepsilon(Py)(Px)z + \varepsilon zy(P^2x) + \varepsilon\delta z(Px)(Py)]$ hence, by (24),

$$\varepsilon\delta P(yz(Px)) = -\delta(Py)(Px)z + \mu\delta zyx + z(Px)(Py). \quad (32)$$

Substituting in the right hand side of (29) the identities (30), (31), (32) by straightforward calculations follows

$$\mu xyz = -\delta x(Pz)(Py) + \delta(Py)(Px)z + (Py)(Pz)x. \quad (33)$$

Substituting $x \to Px, y \to Py, z \to Pz$ in (33) follows $\mu(Px)(Py)(Pz) = -\delta(Px)(P^2z)(P^2y) + \delta(P^2y)(P^2x)(Pz) + (P^2y)(P^2z)(Px)$ hence, by (24), $(Px)(Py)(Pz) = -\mu\delta(Px)zy + \mu\delta yx(Pz) + \mu yz(Px)$, or equivalently

$$(Px)(Py)(Pz) = \mu\varepsilon\delta[-\varepsilon(Px)zy + \varepsilon yx(Pz) + \varepsilon\delta yz(Px)]. \quad (34)$$

Comparing (34) with (28) follows $(Px)(Py)(Pz) = \mu\varepsilon\delta P(xyz)$, or equivalently $P(xyz) = \mu\varepsilon\delta(Px)(Py)(Pz)$. The last line and (34) complete the proof of (27). Moreover, if $\mu\epsilon\delta = 1$ then clearly P is an automorphism of U.
Remark. The identity (25) is a direct consequence of (27) for $\nu = \mu\varepsilon\delta$.
Remark. If U is unitary, it can be also regarded as pseudo unitary by taking $\nu = \mu = 1$ and $P = Id$.

Proposition 2.9 *Let U be an (ε, δ)-FKTS. If U is pseudo unitary with ν, μ and P as before, then $\varepsilon\delta = \mu\nu$ and $K(x, y) = \mu L(y, Px)P - \mu\delta L(x, Py)P$.*

Proof. It follows directly from Proposition 2.8 and Proposition 3.1 of [27].

3 δ-structurable algebras

Our start point is the construction of Lie (super)algebras starting from a class of nonassociative algebras. Hence within the general framework of (ϵ, δ)-FKTSs $(\epsilon, \delta = \pm 1)$ and the standard embedding Lie (super)algebra construction studied in [5],[6],[12]-[14], [23], we define δ-structurable algebras as a class of nonassociative algebras with involution which coincides with the class of structurable algebras for $\delta = 1$ as introduced and studied in [1], [2]. Structurable algebras are a class of nonassociative algebras with involution that include Jordan algebras (with trivial involution), associative algebras with involution, and alternative algebras with involution. They are related to GJTSs 2nd order as introduced and studied in [29], [30] and further studied in [3], [4], [28], [35]-[38], [42]. Their importance lies with constructions of five graded Lie algebras. For $\delta = -1$ the anti-structurable algebras may similarly shed light on the notion of $(-1, -1)$-FKTSs hence (by [5], [6]) on the construction of Lie superalgebras.

Let $(\mathcal{A}, ^-)$ be a finite dimensional nonassociative unital algebra with involution (involutive anti-automorphism, i.e. $\overline{\overline{x}} = x, \overline{xy} = \overline{y}\,\overline{x}, x, y \in \mathcal{A}$) over Φ. The identity element of $\mathcal{A}$ is denoted by 1. Since $\mathrm{char}\Phi \neq 2$, by [1] we have $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$, where $\mathcal{H} = \{a \in \mathcal{A} | \overline{a} = a\}$ and $\mathcal{S} = \{a \in \mathcal{A} | \overline{a} = -a\}$.

Suppose $x, y, z \in \mathcal{A}$. Put $[x, y] := xy - yx$ and $[x, y, z] := (xy)z - x(yz)$. Note that

$$\overline{[x, y, z]} = -[\overline{z}, \overline{y}, \overline{x}]. \tag{35}$$

The operators L_x and R_x are defined by $L_x(y) := xy, R_x(y) := yx$.

For $\delta = \pm 1$ and $x, y \in \mathcal{A}$ define

$$^{\delta}V_{x,y} := L_{L_x(\overline{y})} + \delta(R_x R_{\overline{y}} - R_y R_{\overline{x}}), \tag{36}$$

$$^{\delta}B_{\mathcal{A}}(x,y,z) :=^{\delta} V_{x,y}(z) = (x\overline{y})z + \delta[(z\overline{y})x - (z\overline{x})y], x, y, z \in \mathcal{A}. \tag{37}$$

$^{+}B_{\mathcal{A}}(x,y,z)$ is called the *triple system obtained from the algebra* $(\mathcal{A},^{-})$. We will call $^{-}B_{\mathcal{A}}(x,y,z)$ the *anti-triple system obtained from the algebra* $(\mathcal{A},^{-})$. We shall write for short $V_{x,y} :=^{\delta} V_{x,y}$, $B_{\mathcal{A}} := (^{\delta}B_{\mathcal{A}}, \mathcal{A})$.

A unital non-associative algebra with involution $(\mathcal{A},^{-})$ is called a *structurable algebra* if the following identity is fulfilled

$$[V_{u,v}, V_{x,y}] = V_{V_{u,v}(x),y} - V_{x,V_{v,u}(y)}, \tag{38}$$

for $V_{u,v} = {}^{+}V_{u,v}, V_{x,y} = {}^{+}V_{x,y}, u, v, x, y \in \mathcal{A}$, and we call $(\mathcal{A},^{-})$ an *anti-structurable algebra* if (38) is fulfilled for $V_{u,v} = {}^{-}V_{u,v}, V_{x,y} = {}^{-}V_{x,y}$.

If $(\mathcal{A},^{-})$ is structurable then, in the terminology of [30], the triple system $B_{\mathcal{A}}$ is called a *GJTS* and by [7], $B_{\mathcal{A}}$ is a GJTS of 2nd order.

If $(\mathcal{A},^{-})$ is anti-structurable then we call $B_{\mathcal{A}}$ an *anti-GJTS*.

Put $T_x := V_{x,1}$ for $x \in \mathcal{A}$. Then, by (36), $T_x = L_x + \delta R_{x-\overline{x}}$, for $x \in \mathcal{A}$. In particular, $T_h = L_h$ for $h \in \mathcal{H}$.

Remarks. (i) If $u = h \in \mathcal{H}$ and $x, y \in \mathcal{A}$, (38) becomes

$$[L_h, V_{x,y}] = V_{hx,y} - V_{x,hy}. \tag{39}$$

(ii) Suppose $^{-}$ is the identity map and hence $\mathcal{A}$ is commutative. If $(\mathcal{A},^{-})$ is δ-structurable then $\mathcal{A}$ is a Jordan algebra, by [25]. Conversely, by [32]§3, any Jordan algebra satisfies (39) if $V_{x,y} = {}^{+}V_{x,y}$ for $x, y \in \mathcal{A}$, hence it is structurable. By ([25]), any Jordan algebra is anti-structurable if

$$((hx)y)z - h((xy)z) = (x(yh))z - (xy)(hz) \tag{40}$$

for $h, x, y, z \in \mathcal{A}$. Using commutativity then (40) e.g. can be written $[x, h, y]z = [xy, z, h]$. Clearly, (40) is fulfilled by an associative algebra.

For $s \in \mathcal{S}$ and $h \in \mathcal{H}$ we say that $(\mathcal{A},^{-})$ is $\mathcal{S}$ *skew-alternative* if $[s, x, y] = -[x, s, y]$ while $(\mathcal{A},^{-})$ is $\mathcal{H}$ *skew-alternative* if $[h, x, y] = -[x, h, y]$ for $x, y \in \mathcal{A}$. We shall remark that if $(\mathcal{A},^{-})$ is $\mathcal{S}$ skew-alternative then by [1]§1,

$$[s, x, y] = -[x, s, y] = [x, y, s], s \in \mathcal{S}, x, y \in \mathcal{A}, \tag{41}$$

while if $(\mathcal{A},^-)$ is $\mathcal{H}$ skew-alternative then by (35),

$$[h,x,y] = -[x,h,y] = [x,y,h], h \in \mathcal{H}, x,y \in \mathcal{A}. \tag{42}$$

Proposition 3.1 *([25]) If $(\mathcal{A},^-)$ is structurable, then $(\mathcal{A},^-)$ is $\mathcal{S}$ skew-alternative. If $(\mathcal{A},^-)$ is anti-structurable, then $(\mathcal{A},^-)$ is $\mathcal{H}$ skew-alternative.*

Remark. Let $(\mathcal{A},^-)$ be a δ-structurable algebra and let $\mathrm{Der}(\mathcal{A},^-)$ be the set of derivations of $\mathcal{A}$ that commute with $^-$. By [25], $T_{\mathcal{A}} \cap \mathrm{Der}(\mathcal{A},^-) = 0$ and so we may define the *structure algebra* $\mathrm{Str}(\mathcal{A},^-) := T_{\mathcal{A}} \oplus \mathrm{Der}(\mathcal{A},^-)$ which plays an important role in the study of structurable algebras ([1]) and may play a role in the structure study of anti-structurable algebras.

3.1 Examples

For examples of structurable algebras we refer to [1] and [2].

Remark. Let (B,U) and (B',U') be two triple systems. We say that a linear map μ of U into U' is a *homomorphism* if μ satisfies $\mu(B(x,y,z)) = B'(\mu(x),\mu(y),\mu(z)), x,y,z \in U$. Moreover, if μ is bijective, then μ is called an *isomorphism*. In this case (B,U) and (B',U') are said to be *isomorphic*.

Let $(A,^-)$ be a unital non-associative algebra over Φ with involution $^-$ and let $(A^{op},^-)$ denote the *opposite* algebra, i.e. the algebra with multiplication defined by $x \cdot_{op} y = yx, x,y \in A$, where in the right hand side of the equality the multiplication is done in A. The algebras $(A,^-)$ and $(A^{op},^-)$ are isomorphic under the map $x \mapsto \overline{x}$. Let us define

$${}^{\delta}V^{op}_{x,y} := R_{R_x(\overline{y})} + \delta(L_x L_{\overline{y}} - L_y L_{\overline{x}}), \tag{43}$$

$${}^{\delta}B^{op}_{\mathcal{A}}(x,y,z) :=^{\delta} V^{op}_{x,y}(z) = z(\overline{y}x) + \delta[x(\overline{y}z) - y(\overline{x}z)], x,y,z \in \mathcal{A}. \tag{44}$$

Then $\mathcal{A}$ is a δ-structurable algebra if and only if $\mathcal{A}^{op}$ is a δ-structurable algebra since clearly, B^{op}_A is the triple system obtained from the algebra $(\mathcal{A}^{op},^-)$, and so $B_{\mathcal{A}}$ and $B^{op}_{\mathcal{A}}$ are isomorphic under the map $x \mapsto \overline{x}$, by (37) and (44).

Examples. (i) $\mathcal{M}_{m,n}(\Phi)$ with the product

$$\{x,y,z\} := xy^{\top}z + \delta(zy^{\top}x - zx^{\top}y) \tag{45}$$

where $x, y, z \in \mathcal{M}_{m,n}(\Phi)$, is a $(-1, \delta)$-FKTS ([26]). Hence $\mathcal{M}_{n,n}(\Phi)$ with the involution $x \mapsto x^\top$ is a δ-structurable algebra.

(ii) $\mathcal{M}_{m,n}(\mathbf{C})$ with the product $\{x, y, z\} := x\overline{y}^\top z + \delta(z\overline{y}^\top x - z\overline{x}^\top y)$, where $x, y, z \in \mathcal{M}_{m,n}(\mathbf{C})$, is a $(-1, \delta)$-FKTS ([26]). Hence $\mathcal{M}_{n,n}(\mathbf{C})$ with the involution $x \mapsto \overline{x}^\top$ is a δ-structurable algebra.

Remark. By [23], the following construction of Lie superalgebras is obtained by the standard embedding method. If $U(-1,-1) := \mathcal{M}_{2n,m}(\Phi)$ with the product (45) then the corresponding standard embedding Lie superalgebra is $osp(2n|2m) = D(n,m)$ ([11]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n,2n}(\Phi)$ is $osp(2n|4n)$. Similarly, if $U(-1,-1) := \mathcal{M}_{2n+1,m}(\Phi)$ with the product (45) then the corresponding standard embedding Lie superalgebra is $osp(2n+1|2m) = B(n,m)$ ([11]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n+1,2n+1}(\Phi)$ is $osp(2n+1|4n+2)$.

3.2 Anti-structurable algebras and extended Dynkin diagrams

In this section we give a correspondence between anti-structurable algebras and extended Dynkin diagrams as follows.

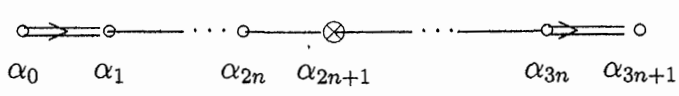
Let $U := \mathcal{M}_{l,l}(\Phi)$ with the product (45) and $\delta = -1$, that is

$$\{x, y, z\} := xy^\top z - zy^\top x + zx^\top y. \tag{46}$$

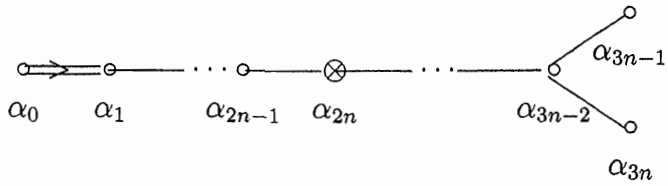
Then we obtain the standard embedding Lie superalgebra as follows; Lie (super)algebra notations and extended Dynkin diagrams are those of [8].

Proposition 3.2 *([26]) Let $(U, \{\ \})$, $U = \mathcal{M}_{l,l}(\Phi)$, be a simple unitary $(-1,-1)$-FKTSs defined by formula (46) and $L(U) = \oplus_{l=-2}^{2} L_l$ be the corresponding standard embedding Lie superalgebra. Then $L(U)$, $L_{-2} \oplus L_0 \oplus L_2$, L_0 and the corresponding extended Dynkin diagrams with $\otimes$ roots deleted are*

$$i) \begin{cases} L(U) = B(n,l) \\ L_{-2} \oplus L_0 \oplus L_2 = C_l \oplus B_n, \\ L_0 = A_{l-1} \oplus B_n \oplus \Phi H \end{cases} \quad \text{for} \quad l = 2n+1,$$



$$ii)\ \begin{cases} L(U) = D(n,l) \\ L_{-2} \oplus L_0 \oplus L_2 = C_l \oplus D_n, \\ L_0 = A_{l-1} \oplus D_n \oplus \Phi H \end{cases} \quad \text{for} \quad l = 2n.$$



Remark. These results mean that study of the correspondence between anti-structurable algebras and extended Dynkin diagrams is an useful concept for the structure theory of triple systems.

References

[1] Allison B.N., A class of nonassociative algebras with involution containing the class of Jordan algebras. Math. Ann. **237** (1978), no. 2, 133-156.

[2] Allison B.N., Models of isotropic simple Lie algebras. Comm. Algebra **7** (1979), no. 17, 1835-1875.

[3] Asano H.; Yamaguti K., A construction of Lie algebras by generalized Jordan triple systems of second order. Nederl. Akad. Wetensch. Indag. Math. **42** (1980), no. 3, 249-253.

[4] Asano H., Classification of non-compact real simple generalized Jordan triple systems of the second kind. Hiroshima Math. J. **21** (1991), 463-489.

[5] Elduque A.; Kamiya N.; Okubo S., Simple $(-1,-1)$ balanced Freudenthal Kantor triple systems. Glasg. Math. J. **11** (2003), no. 2, 353-372.

[6] Elduque A.; Kamiya N.; Okubo S., $(-1,-1)$ balanced Freudenthal Kantor triple systems and noncommutative Jordan algebras. J. Algebra **294** (2005), no. 1, 19-40.

[7] Faulkner J.R., Structurable triples, Lie triples, and symmetric spaces. Forum Math. **6** (1994), 637-650.

[8] Frappat L.; Sciarrino A.; Sorba P., *Dictionary on Lie Algebras and Superalgebras*, Academic Press, San Diego, California 92101-4495, USA, 2000, xxii+410 pp.

[9] Jacobson N., Lie and Jordan triple systems, Amer. J. Math. **71** (1949), 149-170.

[10] Jacobson N., *Structure and representations of Jordan algebras.* Amer. Math. Soc. Colloq. Publ., Vol. XXXIX Amer. Math. Soc., Providence, R.I. 1968, x+453 pp.

[11] Kac V.G., Lie superalgebras. Advances in Math. **26** (1977), no. 1, 8-96.

[12] Kamiya N., A structure theory of Freudenthal-Kantor triple systems. J. Algebra **110** (1987), no. 1, 108-123.

[13] Kamiya N., A construction of anti-Lie triple systems from a class of triple systems, Mem. Fac. Sci. Shimane Univ. **22** (1988), 51-62.

[14] Kamiya N., A structure theory of Freudenthal-Kantor triple systems. II. Comment. Math. Univ. St. Paul. **38** (1989), no. 1, 41-60.

[15] Kamiya N., On (ϵ, δ)-Freudenthal-Kantor triple systems. *Nonassociative algebras and related topics* (Hiroshima, 1990), 65-75. World Sci. Publ., River Edge, NJ, 1991.

[16] Kamiya N. The construction of all simple Lie algebras over $\mathbf{C}$ from balanced Freudenthal-Kantor triple systems. *Contributions to general algebra*, 7, 205-213, Hölder-Pichler-Tempsky, Vienna, 1991.

[17] Kamiya N. On Freudenthal-Kantor triple systems and generalized structurable algebras. *Nonass. alg. and its appl.* (Oviedo, 1993), 198-203, Math. Appl., 303, Kluwer Acad. Publ., Dordrecht, 1994.

[18] Kamiya N., On the Peirce decompositions for Freudenthal-Kantor triple systems. Comm. Algebra **25** (1997), no. 6, 1833-1844.

[19] Kamiya N., On a realization of the exceptional simple graded Lie algebras of the second kind and Freudenthal-Kantor triple systems. Bull. Polish Acad. Sci. Math. 46 (1998), no. 1, 55-65.

[20] Kamiya N.; Okubo S., On δ-Lie supertriple systems associated with $(\epsilon, \delta)-$ Freudenthal-Kantor supertriple systems. Proc. Edinburgh Math. Soc. **43** (2000), no. 2, 243-260.

[21] Kamiya N.; Okubo S., Construction of Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ from some triple systems. Proc. Edinburgh Math. Soc. **46** (2003), no. 1, 87-98.

[22] Kamiya N.; Okubo S., On generalized Freudenthal-Kantor triple systems and Yang-Baxter equations, *Proc. XXIV Inter. Colloq. Group Theoretical Methods in Physics*, IPCS, vol. 173 (2003), 815-818.

[23] Kamiya N.; Okubo S., A construction of simple Lie superalgebras of certain types from triple systems. Bull. Austral. Math. Soc. **69** (2004), no. 1, 113-123.

[24] Kamiya N.; Okubo S., Composition, quadratic, and some triple systems. *Non-associative algebra and its applications*, 205-231, Lect. Notes Pure Appl. Math., 246, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[25] Kamiya N.; Mondoc D., A new class of nonassociative algebras with involution. Proc. Japan Acad. Ser. A **84** (2008), no. 5, 68-72.

[26] Kamiya N.; Mondoc D.; Okubo S., A structure theory of $(-1,-1)$-Freudenthal Kantor triple systems. Bull. Australian Math. Soc. **81** (2010), 132-155.

[27] Kamiya N.; Mondoc D.; Okubo S., On Lie and Jordan structures associated with (ε,δ)-Freudenthal Kantor triple systems, Mitt. Math. Gesells. Hamburg **29** (2010), 109-123.

[28] Kaneyuki S.; Asano H., Graded Lie algebras and generalized Jordan triple systems. Nagoya Math. J. **112** (1988), 81-115.

[29] Kantor I.L., Graded Lie algebras. Trudy Sem. Vect. Tens. Anal. **15** (1970), 227-266.

[30] Kantor I.L., Some generalizations of Jordan algebras. Trudy Sem. Vect. Tens. Anal. **16** (1972), 407-499.

[31] Kantor I.L. Models of exceptional Lie algebras. Soviet Math. Dokl. Vol. **14** (1973), no. 1, 254-258.

[32] Koecher M., Embedding of Jordan algebras into Lie algebras I. II., Amer. J. Math **89** (1967), 787-816 and **90** (1968), 476-510.

[33] Lister W.G., A structure theory of Lie triple systems, Trans. Amer. Math. Soc. **72** (1952), 217-242.

[34] Meyberg K., Lectures on algebras and triple systems, *Lecture Notes*, The University of Virginia, Charlottesville, Va., 1972, v+226 pp.

[35] Mondoc D., Models of compact simple Kantor triple systems defined on a class of structurable algebras of skew-dimension one. Comm. Algebra **34** (2006), no. 10, 3801-3815.

[36] Mondoc D., On compact realifications of exceptional simple Kantor triple systems. J. Gen. Lie Theory Appl. **1** (2007), no. 1, 29-40.

[37] Mondoc D., Compact realifications of exceptional simple Kantor triple systems defined on tensor products of composition algebras. J. Algebra **307** (2007), no. 2, 917-929.

[38] Mondoc D., Compact exceptional simple Kantor triple systems defined on tensor products of composition algebras. Comm. Algebra **35** (2007), no. 11, 3699-3712.

[39] Okubo S., Introduction to octonion and other non-associative algebras in physics. *Montroll Memorial Lecture Series in Mathematical Physics, 2.* Cambridge University Press, Cambridge, 1995, xii+136 pp.

[40] Okubo S.; Kamiya N., Jordan-Lie superalgebra and Jordan-Lie triple system. J. Algebra **198** (1997), no. 2, 388-411.

[41] Okubo S.; Kamiya N., Quasi-classical Lie superalgebras and Lie supertriple systems. Comm. Algebra **30** (2002), no. 8, 3825-3850.

[42] Okubo S., Symmetric triality relations and structurable algebras. Linear Algebra Appl. **396** (2005), 189-222.

[43] Scheunert M., The theory of Lie superalgebras. An introduction. *Lecture Notes in Mathematics*, 716. Springer, Berlin, 1979, x+271 pp.

[44] Tits J., Une classe d'algèbres de Lie en relation avec les algèbres de Jordan. Nederl. Acad. Wetensch. Proc. Ser. A **65** (1962), 530-535.

[45] Yamaguti K.; Ono A. On representations of Freudenthal-Kantor triple systems $U(\epsilon, \delta)$, Bull. Fac. School Ed. Hiroshima Univ., **7** (1984), no. II, 43-51.

ALGEBRAS GROUPS AND GEOMETRIES 35 283 -318 (2018) - 283 -
DOI 10.29083/AGG.35.03.2018

FOUR INDEPENDENT EIGENVECTORS AND THEIR PROPERTIES FOR THE PROBLEM OF ACOUSTIC 4P-SH-WAVE PROPAGATION INCORPORATING GRAVITATIONAL PHENOMENA

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Received October 21, 2019

Abstract

This theoretical investigation has obtained four independent pairs of main eigenvectors called the *E*-, *H*-, *G*-, and *F*-eigenvectors. Their properties are also demonstrated. The obtained eigenvectors together with the eigenvalues represent the solutions of the coupled equations of motion when the shear-horizontal (SH) acoustic wave propagation in the 6 *mm* solids is coupled with the electrical, magnetic, gravitational, and cogravitational potentials, i.e. 4P-SH-waves. To obtain suitable forms of the eigenvectors is very important because they form the final solutions of the propagation velocities, with which the acoustic waves propagate in the solids. It is expected that their proper forms can demonstrate dominating factors causing the acoustic wave propagation, i.e. the physical properties of the wave processes. To know the acoustic wave propagation velocities in the solids is one of very important engineering problems in the signal processing technology. This allows the constitution of various technical devices such as filters, sensors, delay lines, energy harvesting and wireless devices, etc. The successful development of the theoretical, mathematical, experimental, engineering, and technological directions of investigations of the four-potential waves and their applications can result in the instant interplanetary (interstellar and even intergalactic) communications, for instance, the instant interplanetary Internet. This is possible because the evaluated speeds of the recently discovered fast four-potential waves propagating in a vacuum can be thirteen orders faster than the speed of light, i.e. the speed of the electromagnetic waves representing two-potential waves.

Keywords: Transversely isotropic continuous media; Acoustic wave propagation; Gravitational effects; Four potential coupling problem.

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1 Introduction

In the jubilee year 2016, it was developed the theory [1] that incorporates the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems in the wave motion process. As a result, the new four-potential shear-horizontal surface acoustic wave (4P-SH-SAW) [1] was discovered when the acoustic wave propagates in the transversely isotropic material of symmetry class 6 *mm*. It was a centenary jubilee of the prediction of the existence of the gravitational waves by Albert Einstein in 1916 [2]. To this jubilee, a very large international research team has experimentally proved the existence of the gravitational waves. Their experimental results [3] published in 2016 have soundly confirmed the existence of the theoretically predicted gravitational waves.

Developed theory [1] uses this experimental fact of the propagation of the gravitational waves in the free space (vacuum) at the speed of light. It is well-known that the electromagnetic waves also propagate in a vacuum at the speed of light. The speed of light was precisely measured only in the 1970s and now can be found in any modern reference book on physics. For instance, the measured value of the speed of light can be found in popular reference book [4]. So, it is possible to write down the following value of the speed of light: $C_L = (\varepsilon_0\mu_0)^{-1/2} \sim (\gamma_0\eta_0)^{-1/2} = 2.99792458 \times 10^8$ [m/s], where ε_0, μ_0, γ_0, and η_0 are the electric constant, magnetic constant, gravitic constant, and cogravitic constant for a vacuum, respectively. A gravitational and electromagnetic analogy was first stated by Heaviside [5] and then developed by many researchers. The famous book by Jefimenko [6] operates with gravitation and cogravitation to develop Newton's theory of gravitation to its physical and mathematical conclusion. So, the purely gravitational analogy to the theory of electromagnetism is now called the theory of the gravitoelectromagnetism. It is expected that the gravitic constant γ_0 and the cogravitic constant η_0 for a vacuum can be precisely measured in this century and then recorded in physics reference books.

Created theory [1] relating to numerical experiment takes into account the fact that the speed of the electromagnetic waves in both a solid and a vacuum is five orders faster than the speed of any acoustic wave in a solid. As a result, a quasi-static approximation is used in the theory of acoustic wave propagation in continuous solid media. In the solids, the slow speed of the new 4P-SH-SAW [1] can naturally depend on the speed of the electromagnetic wave. However, it also depends on the speed of new fast waves that can be thirteen orders faster

than the speed of light. These new fast waves can also propagate in a vacuum with the following speeds: $\Lambda_{01} = (\zeta_0\lambda_0)^{-1/2} \to 10^{13}C_L$ and $\Lambda_{02} = (\xi_0\beta_0)^{-1/2} \to 10^{13}C_L$, where the vacuum parameters ζ_0, λ_0, ξ_0, and β_0 are called the gravitoelectric, cogravitomagnetic, cogravitoelectric, and gravitomagnetic constants, respectively. These vacuum parameters relating to the gravitational phenomena must be also measured and recorded in any modern reference book on physics.

The existence of the new fast waves propagating in a vacuum with the speeds Λ_{01} and Λ_{02} can be true because the famous Russian astronomer and astrophysicist Kozyrev [7] has created his own tools, with which he has observed true positions of stars from his observatory. It is obvious that astronomers use optic tools to observe stars. As a result, they can for instance observe a star distant from the Earth on several light years at its position it was several light years ago. There are a lot of stars distant from the Earth on 10, 100, 1000 light years. Therefore, any optic tools for observation of stars cannot record true positions of the stars. To observe the true position of any star, Kozyrev [7] has first used some optic tools to observe a star. After that, he has calculated the true position of the star and used his own tools to observe this star at the true position. This means that there is something that can propagate in a vacuum significantly faster than the speed of light. To explain this phenomenon, Kozyrev has developed his own theory called the theory of time.

Kozyrev is not the single researcher who has found that there is something significantly faster than the speed of light. The French mathematician Pierre-Simon Laplace [8], also known as the French Newton, has evaluated the stability of the Solar system. He has investigated the motion of space bodies in the Solar system and concluded that gravity must propagate millions of times faster than light, since otherwise violation of Newton's law of universal gravitation would be observed. It is worth noting that Newton's law of universal gravitation assumes an infinitely large speed of gravity, i.e. the gravity can instantly propagate. However, the first attempt to explain Newton's law of gravitation was done by Nicolas Fatio de Duillier in 1690 and by Georges-Louis Le Sage in 1748. This is a kinetic theory of gravity originally proposed by Fatio and later developed by Le Sage. The theory proposed a mechanical explanation for Newton's gravitational force in terms of streams of tiny unseen particles impacting all material objects from all directions [9]. Concerning the movements of planets in the Solar system in the fluid consisting of the particles (i.e. in the

free space), Le Sage [9] has used an argument to demonstrate that the speed of the fluid must be thirteen orders faster than the speed of light. To review all existing evaluations of the speed of gravity and possible gravitational phenomena does not represent the main purpose of this study. Therefore, it is necessary to return to modern theory [1] because the thermodynamic functions and the thermodynamic variables were used to develop this theory.

The hottest application of the new fast waves propagating in a vacuum at the speeds A_{01} and A_{02} is the instant interplanetary communication [10, 11] inside the Solar system, for instance, the space Internet. The possible applications proposed in paper [10] for the new fast waves propagating in the continuous medium such as a vacuum is the natural consequence from theory [1] developed for the continuous medium such as a solid. Theory [1] considers the acoustic wave propagation in the smart solid materials such as the piezoelectromagnetics (magnetoelecroelastics) with taking into account the gravitational phenomena. In this case, the gravitational phenomena significantly complicate the theoretical developments compared with the case of pure piezoelectromagnetics without the phenomena [12, 13, 14]. However, it is necessary to treat the gravitational phenomena because the propagation of acoustic waves means energy transmission between the wave generator and the wave detector. One century ago Einstein has postulated that any kind of energy (and any change in energy) is coupled with gravitation. Therefore, theory [1] treats the gravitational and cogravitational subsystems in addition to the mechanical, electrical, and magnetic subsystems. As a result, the wave motions are caused by exchange interactions (i.e. some energetic exchanges) among the subsystems. It is expected that such energetic system consisting of five subsystems can be more reach than the purely piezoelectromagnetic system consisting of only three subsystems such as the mechanical, electrical, and magnetic subsystems. It is natural because piezoelectromagnetics with (without) taking into account the gravitational phenomena can be used instead of pure piezoelectrics.

Today, piezoelectrics are utilized in various technical devices such as filters, (chemical and biological) sensors, delay lines, energy harvesting and wireless devices, etc. They are also used in the cutting-edge technologies of mobile communication and wireless sensors [15, 16]. The piezoelectric materials are also studied for applications in energy harvesting devices [16, 17, 18, 19]. However, smart piezoelectromagnetic materials are more preferable for constitution of the wireless devices [20, 21, 22] in comparison with the

conventional piezoelectrics. It is also expected that the piezogravitocogravitoelectromagnetic materials [1] with the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems can be more reach regarding the energy harvesting and energy conversion. Recently, Füzfa [23] has found a method to study interactions of the electrical or magnetic subsystem with the gravitational or cogravitational subsystem at the laboratory conditions on the Earth instead of very expensive space experiments. The theoretical, mathematical, experimental, engineering, and many relative investigations can be useful for the development of infrastructure including the technical devices for generating, receiving, and signal processing of the new fast waves for the new era of the instant interplanetary communication.

The theoretical investigations were initiated in pioneer paper [1]. However, this created theory has released some mathematical problems that must be also resolved. One of the most important mathematical problems is the finding of proper eigenvectors because their forms can result in final solutions for the propagating velocities of the acoustic waves. Theoretical work [24] has demonstrated only half of the twenty-four possible eigenvectors that can be obtained in the framework of theory [1]. However, all of them can be reduced to four main independent eigenvectors. To obtain these four eigenvectors called the *E*-, *H*-, *G*-, and *F*-eigenvectors and their properties is the main purpose of this study. The theoretical part and mathematical methods are very important even in the case of the acoustic wave propagation in the pure piezoelectrics. To find propagation velocities in different structures represents an engineering problem. There is famous book by Dieulesaint and Royer [25] concerning applications of the elastic waves in the piezoelectric solids to signal processing. Therefore, original theory [1] represents a rudiment for further development of the theory of the acoustic wave propagation in the piezogravitocogravitoelectromagnetic solids guided by the surface [1], interface [26], and plate (thin film) [27]. In all these cases, it is very important to know eigenvalues and eigenvectors representing the solutions of the corresponding coupled equations of motion.

2 The equations of motion and solutions

To construct the coupled equations of motion, it is necessary first of all to treat the suitable thermodynamic functions and thermodynamic variables for an adiabatic process in solid continuous media. For the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems, the thermodynamic

functions are the mechanical stress, electrical induction, magnetic induction, gravitational induction, and cogravitational induction. The thermodynamic variables are the mechanical strain, electrical field, magnetic field, gravitational field, and cogravitational field. So, this is a multidisciplinary research arena. The mechanical parameters came from the theory of elasticity, the electrical and magnetic parameters from the theory of electromagnetism, and the gravitational and cogravitational parameters from the theory of gravitoelectromagnetism. There is some analogy between the theories of gravitation and the electromagnetism [5]. As a result, some analogy can be found between the theories of electricity and gravitation (i.e. gravitoelectricity) and between the theories of magnetism and cogravitation [6] (i.e. gravitomagnetism [28]). This part of theory is quite large and the reader can find it in paper [1]. However, it is also necessary to mention that the quasi-static approximation must be applied to obtain the coupled equations of motion in the form of a set of second-order partial differential equations. This approximation is used because the speed of the electromagnetic waves in a solid is close to the speed of light in a vacuum and five orders faster than the speed of any acoustic wave.

To resolve the second-order partial differential equations, it is natural to exploit the plane wave solutions. These solutions naturally transform the differential form of the coupled equations of motion into the tensor form of the equations. In the common case of the acoustic wave propagation in an anisotropic solid, the tensor form can be written down in the following compact expression called the modified Green-Christoffel equation [1]: $\left(GL_{IJ} - \delta_{IJ}\rho V_{ph}\right)U_I^0 = 0$, where the indices I and J run from 1 to 7 and ρ is the mass density listed in table 1. The phase velocity V_{ph} will be defined below by expression (7). GL_{IJ} stands for the components of the modified symmetric tensor [1] and δ_{IJ} represents the Kronecker delta-function with the following conditions: $\delta_{IJ} = 1$ for $I = J < 4$, $\delta_{IJ} = 0$ for $I \neq J$, and $\delta_{44} = \delta_{55} = \delta_{66} = \delta_{77} = 0$. Also, parameters U_I^0 represent the components of the eigenvector $(U_1^0, U_2^0, U_3^0, U_4^0, U_5^0, U_6^0, U_7^0)$. This tensor form of the coupled equations of motion represents the common problem for determination of the eigenvalues and eigenvectors. This common problem is very complicated and can be numerically resolved. However there are the high-symmetry propagation directions in solid materials with certain crystal symmetries [25, 29] that allow significant simplifications leading to some analytical solutions.

Table 1: The material parameters of the transversely isotropic material (crystal) of symmetry class 6 *mm* and their physical dimensions.

Material parameter	**Symbol [dimension]**
Mass density	ρ [kg/m^3]
Elastic stiffness constant	$C = C_{44} = C_{66}$ [kg/(m×s^2)], [N/m^2]
Piezoelectric constant	$e = e_{16} = e_{34}$ [kg$^{1/2}$/m$^{3/2}$], [C/m^2]
Piezomagnetic coefficient	$h = h_{16} = h_{34}$ [kg$^{1/2}$/(m$^{1/2}$×s)], [N/(A×m)], [T]
Piezogravitic constant	$g = g_{16} = g_{34}$ [kg/m^2]
Piezocogravitic coefficient	$f = f_{16} = f_{34}$ [s^{-1}]
Electric constant	$\varepsilon = \varepsilon_{11} = \varepsilon_{33}$ [s^2/m^2], [F/m]
Magnetic constant	$\mu = \mu_{11} = \mu_{33}$ [-], [N×s^2/C^2], [H/m], [N/A^2], [T×m/A], [Wb/(A×m)], [V×s/(A×m)]
Electromagnetic constant	$\alpha = \alpha_{11} = \alpha_{33}$ [s/m], [N×s/(V×C)]
Gravitic constant	$\gamma = \gamma_{11} = \gamma_{33}$ [kg×s^2/m^3], [kg^2/(N×m^2)]
Cogravitic constant	$\eta = \eta_{11} = \eta_{33}$ [m/kg]
Gravitocogravitic constant	$\vartheta = \vartheta_{11} = \vartheta_{33}$ [s/m]
Gravitoelectric constant	$\zeta = \zeta_{11} = \zeta_{33}$ [kg$^{1/2}$×s^2/m$^{5/2}$], [C×kg/(J×m)]
Cogravitoelectric constant	$\xi = \xi_{11} = \xi_{33}$ [s/(kg$^{1/2}$×m$^{1/2}$)], [m/Wb]
Gravitomagnetic constant	$\beta = \beta_{11} = \beta_{33}$ [kg$^{1/2}$×s/m$^{3/2}$], [T×kg×m/J]
Cogravitomagnetic constant	$\lambda = \lambda_{11} = \lambda_{33}$ [m$^{1/2}$/kg$^{1/2}$], [T×m^3/(C×Wb)]

For a solid of symmetry class 6 *mm*, the high-symmetry propagation directions on the certain cuts of the solid are well-known. According to review paper [30], the sixfold symmetry axis must be directed along the x_2-axis, perpendicular to both the wave propagation direction towards the x_1-axis and the surface normal along the x_3-axis. The rectangular coordinate system $\{x_1, x_2, x_3\}$ and the corresponding wavevector components $\{k_1, k_2, k_3\}$ are used. For this case, the modified Green-Christoffel equation representing a set of seven homogeneous equations splits into two independent sets. The first set of two homogeneous equations with the eigenvector (U_1^0, U_3^0) corresponds to the purely mechanical wave with the in-plane polarization. The second set of five homogeneous equations with the eigenvector $(U_2^0, U_4^0, U_5^0, U_6^0, U_7^0)$ represents the

main interest in this study because here the acoustic wave propagation is coupled with the following four potentials: the electrical φ, magnetic ψ, gravitational Φ, and cogravitational Ψ potentials. This is the case of propagation of the anti-plane polarized four-potential shear-horizontal (4P-SH) acoustic waves.

For this case, it is possible to deal only with the suitable *GL*-tensor components of the modified Green-Christoffel equation [1]. Therefore, the coupled equations of motion describing the 4P-SH-wave propagation coupled with the four potentials can be then expressed by the following five homogeneous equations:

$$\begin{pmatrix} GL_{22}-\rho V_{ph}^2 & GL_{24} & GL_{25} & GL_{26} & GL_{27} \\ GL_{42} & GL_{44} & GL_{45} & GL_{46} & GL_{47} \\ GL_{52} & GL_{54} & GL_{55} & GL_{56} & GL_{57} \\ GL_{62} & GL_{64} & GL_{65} & GL_{66} & GL_{67} \\ GL_{72} & GL_{74} & GL_{75} & GL_{76} & GL_{77} \end{pmatrix} \begin{pmatrix} U_2^0 \\ U_4^0 \\ U_5^0 \\ U_6^0 \\ U_7^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

With the material parameters listed in table 1, equations (1) must be rewritten as follows:

$$\begin{pmatrix} C\left[m-\left(V_{ph}/V_{t4}\right)^2\right] & em & hm & gm & fm \\ em & -\varepsilon m & -\alpha m & -\zeta m & -\xi m \\ hm & -\alpha m & -\mu m & -\beta m & -\lambda m \\ gm & -\zeta m & -\beta m & -\gamma m & -\vartheta m \\ fm & -\xi m & -\lambda m & -\vartheta m & -\eta m \end{pmatrix} \begin{pmatrix} U^0 \\ \phi^0 \\ \psi^0 \\ \Phi^0 \\ \Psi^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2}$$

where

$$m = 1 + n_3^2 \tag{3}$$

$$n_3 = k_3/k \tag{4}$$

$$(U^0, \phi^0, \psi^0, \Phi^0, \Psi^0) = (U_2^0, U_4^0, U_5^0, U_6^0, U_7^0) \tag{5}$$

$$V_{t4} = \sqrt{C/\rho} \tag{6}$$

$$V_{ph} = \omega / k \tag{7}$$

All the suitable eigenvalues n_3 and the corresponding eigenvectors respectively defined by expressions (4) and (5) must be obtained by resolving equation (2). The velocity V_{t4} defined by expression (6) represents the speed of the purely mechanical bulk SH-wave. The phase velocity V_{ph} defined by expression (7) is proportional to the angular frequency ω and inversely proportional to the wavenumber k in the propagation direction.

Expanding the determinant of the coefficient matrix in equations (2), it is possible to obtain a secular equation representing a polynomial that must be equal to zero. To find all the possible polynomial roots n_3, it is convenient to rewrite the polynomial in the form of several suitable cofactors in order that each cofactor contains two polynomial roots n_3. Therefore, it is possible to write down the following convenient form consisting of five cofactors:

$$m \times m \times m \times m \times \det(V_{ph}) = 0 \tag{8}$$

where

$$\det(V_{ph}) = \begin{vmatrix} C\left[m - \left(V_{ph}/V_{t4}\right)^2\right] & e & h & g & f \\ em & -\varepsilon & -\alpha & -\zeta & -\xi \\ hm & -\alpha & -\mu & -\beta & -\lambda \\ gm & -\zeta & -\beta & -\gamma & -\vartheta \\ fm & -\xi & -\lambda & -\vartheta & -\eta \end{vmatrix} \tag{9}$$

It is clearly seen that the first four cofactors in equation (8) are identical. Therefore, they actually release the following four pairs of identical eigenvalues:

$$n_3^{(1,2)} = n_3^{(3,4)} = n_3^{(5,6)} = n_3^{(7,8)} = \pm j \tag{10}$$

where $j = \sqrt{-1}$ is the imaginary unity.

Expanding determinant (9) leads to the following fifth pair of the eigenvalues:

$$n_3^{(9,10)} = \pm j\sqrt{1 - \left(V_{ph}/V_{temgc}\right)^2} \tag{11}$$

where

$$V_{temgc} = \sqrt{C/\rho}\left(1 + K_{emgc}^2\right)^{1/2} \tag{12}$$

$$K_{emgc}^2 = \frac{A_1}{A_2} \tag{13}$$

$$A_1 = e^2(\mu\gamma\eta + 2\beta\lambda\vartheta - \lambda^2\gamma - \beta^2\eta - \vartheta^2\mu) + h^2(\varepsilon\gamma\eta + 2\zeta\xi\vartheta - \vartheta^2\varepsilon - \zeta^2\eta - \xi^2\gamma) + g^2(\varepsilon\mu\eta + 2\alpha\xi\lambda - \lambda^2\varepsilon - \alpha^2\eta - \xi^2\mu) + f^2(\varepsilon\mu\gamma + 2\alpha\beta\zeta - \beta^2\varepsilon - \alpha^2\gamma - \zeta^2\mu) + 2eh(\vartheta^2\alpha + \zeta\beta\eta + \xi\gamma\lambda - \alpha\gamma\eta - \zeta\lambda\vartheta - \xi\beta\vartheta) + 2eg(\alpha\beta\eta + \lambda^2\zeta + \xi\vartheta\mu - \alpha\lambda\vartheta - \zeta\mu\eta - \xi\beta\lambda) + 2ef(\alpha\gamma\lambda + \zeta\vartheta\mu + \beta^2\xi - \alpha\beta\vartheta - \zeta\beta\lambda - \xi\mu\gamma) + 2hg(\varepsilon\lambda\vartheta + \zeta\alpha\eta + \xi^2\beta - \varepsilon\eta\beta - \zeta\lambda\xi - \xi\vartheta\alpha) + 2hf(\varepsilon\beta\vartheta + \zeta^2\lambda + \xi\alpha\gamma - \varepsilon\lambda\gamma - \zeta\vartheta\alpha - \xi\zeta\beta) + 2gf(\varepsilon\beta\lambda + \alpha^2\vartheta + \xi\mu\zeta - \varepsilon\mu\vartheta - \alpha\zeta\lambda - \alpha\beta\xi) \tag{14}$$

$$A_2 = C(\varepsilon\mu - \alpha^2)(\gamma\eta - \vartheta^2) + C(\beta^2\xi^2 - \xi^2\mu\gamma - \beta^2\varepsilon\eta) + C(\lambda^2\zeta^2 - \lambda^2\varepsilon\gamma - \zeta^2\mu\eta) + 2C(\gamma\alpha\xi\lambda + \eta\alpha\beta\zeta + \varepsilon\beta\lambda\vartheta + \mu\zeta\xi\vartheta - \zeta\xi\beta\lambda - \alpha\zeta\lambda\vartheta - \alpha\beta\xi\vartheta) \tag{15}$$

Table 2: The physical dimensions of factors M_{ee}, M_{hh}, M_{gg}, M_{ff}, M_{eh}, M_{eg}, M_{ef}, M_{hg}, M_{hf}, M_{gf} and CEMGCMC coupling mechanisms M_1, M_2, M_3, M_4, M_5.

Factor	Dimension
M_{ee}	s^2/m^2
M_{hh}	s^4/m^4
M_{gg}	$s^2/(kg\times m)$
M_{ff}	$kg\times s^4/m^5$
M_{eh}	s^3/m^3
M_{eg}	$s^2/(kg^{1/2}\times m^{3/2})$
M_{ef}	$kg^{1/2}\times s^3/m^{7/2}$
M_{hg}	$s^3/(kg^{1/2}\times m^{5/2})$
M_{hf}	$kg^{1/2}\times s^4/m^{9/2}$
M_{gf}	s^3/m^3
M_1	$kg^{1/2}\times s^2/m^{7/2}$
M_2	$kg^{1/2}\times s^3/m^{9/2}$
M_3	s^2/m^3
M_4	$kg\times s^3/m^5$
M_5	s^4/m^4
eM_1, hM_2, gM_3, fM_4, CM_5	$kg\times s^2/m^5$

In definition (12), the velocity V_{iemgc} of the shear-horizontal bulk acoustic wave (SH-BAW) is coupled with the electrical φ, magnetic ψ, gravitational Φ, and cogravitational Ψ potentials. This velocity depends on the nondimensional parameter K^2_{emgc} (13) called the coefficient of the electromagnetogravitocogravitomechanical coupling (CEMGCMC). It is useful to rewrite the parameter K^2_{emgc} (13) as a function of the CEMGCMC coupling mechanisms M_1, M_2, M_3, M_4, and M_5. Their physical dimensions are listed in table 2. So, the parameter K^2_{emgc} reads:

$$K^2_{emgc} = \frac{eM_1+hM_2+gM_3+fM_4}{CM_5} \tag{16}$$

where

$$M_1 = eM_{ee} + hM_{eh} + gM_{eg} + fM_{ef} \tag{17}$$

$$M_2 = eM_{eh} + hM_{hh} + gM_{hg} + fM_{hf} \tag{18}$$

$$M_3 = eM_{eg} + hM_{hg} + gM_{gg} + fM_{gf} \tag{19}$$

$$M_4 = eM_{ef} + hM_{hf} + gM_{gf} + fM_{ff} \tag{20}$$

$$\begin{aligned} M_5 = \varepsilon M_{ee} + \alpha M_{eh} + \zeta M_{eg} + \xi M_{ef} = \alpha M_{eh} + \mu M_{hh} + \beta M_{hg} + \lambda M_{hf} = \\ \zeta M_{eg} + \beta M_{hg} + \gamma M_{gg} + \vartheta M_{gf} = \xi M_{ef} + \lambda M_{hf} + \vartheta M_{gf} + \eta M_{ff} = \\ (\varepsilon\mu - \alpha^2)(\gamma\eta - \vartheta^2) + (\beta\xi - \lambda\zeta)^2 - (\xi^2\mu\gamma + \beta^2\varepsilon\eta + \lambda^2\varepsilon\gamma + \zeta^2\mu\eta) + \\ 2(\gamma\alpha\xi\lambda + \eta\alpha\beta\zeta + \varepsilon\beta\lambda\vartheta + \mu\zeta\xi\vartheta - \alpha\zeta\lambda\vartheta - \alpha\beta\xi\vartheta) \end{aligned} \tag{21}$$

because

$$M_{ee} = \mu\gamma\eta + 2\beta\lambda\vartheta - \lambda^2\gamma - \beta^2\eta - \vartheta^2\mu \tag{22}$$

$$M_{hh} = \varepsilon\gamma\eta + 2\zeta\xi\vartheta - \vartheta^2\varepsilon - \zeta^2\eta - \xi^2\gamma \tag{23}$$

$$M_{gg} = \varepsilon\mu\eta + 2\alpha\xi\lambda - \lambda^2\varepsilon - \alpha^2\eta - \xi^2\mu \tag{24}$$

$$M_{ff} = \varepsilon\mu\gamma + 2\alpha\beta\zeta - \beta^2\varepsilon - \alpha^2\gamma - \zeta^2\mu \tag{25}$$

$$M_{eh} = \zeta\beta\eta + \xi\gamma\lambda + \vartheta^2\alpha - \alpha\gamma\eta - \zeta\lambda\vartheta - \xi\beta\vartheta \tag{26}$$

$$M_{eg} = \alpha\beta\eta + \xi\vartheta\mu + \lambda^2\zeta - \alpha\lambda\vartheta - \zeta\mu\eta - \xi\beta\lambda \tag{27}$$

$$M_{ef} = \alpha\gamma\lambda + \zeta\vartheta\mu + \beta^2\xi - \alpha\beta\vartheta - \zeta\beta\lambda - \xi\mu\gamma \tag{28}$$

$$M_{hg} = \varepsilon\lambda\vartheta + \zeta\alpha\eta + \xi^2\beta - \alpha\xi\vartheta - \zeta\lambda\xi - \varepsilon\eta\beta \tag{29}$$

$$M_{hf} = \varepsilon\beta\vartheta + \xi\alpha\gamma + \zeta^2\lambda - \alpha\zeta\vartheta - \zeta\xi\beta - \varepsilon\lambda\gamma \tag{30}$$

$$M_{gf} = \varepsilon\beta\lambda + \xi\mu\zeta + \alpha^2\vartheta - \alpha\zeta\lambda - \alpha\beta\xi - \varepsilon\mu\vartheta \tag{31}$$

The physical dimensions of parameters (22)-(31) are listed in table 2. To obtain the eigenvector components, it is necessary to resolve equation (2). This was done in paper [24]. However, paper [24] has demonstrated only half of twenty-four possible eigenvectors. Moreover, each of the twenty-four eigenvectors can be reduced to one of four main independent eigenvectors that will be obtained together with their properties in the next four sections. These independent eigenvectors are called the *F*-eigenvectors, *G*-eigenvectors, *H*-eigenvectors, and *E*-eigenvectors. In conclusion to this section, it is useful to write down some useful expressions for the nondimensional material parameters defined below by equalities from (32) to (41). These material parameters are used in the following four sections. For them, it is necessary to mention that the coefficient K^2_{emgc} (CEMGCMC) is defined by expression (16) and all the material constants are listed in table 1. These nondimensional material parameters read:

$$K_E = 1 - \frac{K_e^2}{K^2_{emgc}}, K_e^2 = \frac{e^2}{C\varepsilon} \tag{32}$$

$$K_M = 1 - \frac{K_m^2}{K^2_{emgc}}, K_m^2 = \frac{h^2}{C\mu} \tag{33}$$

$$K_G = 1 - \frac{K_g^2}{K^2_{emgc}}, K_g^2 = \frac{g^2}{C\gamma} \tag{34}$$

$$K_F = 1 - \frac{K_f^2}{K_{emgc}^2}, K_f^2 = \frac{f^2}{C\eta} \tag{35}$$

$$K_A = 1 - \frac{K_\alpha^2}{K_{emgc}^2}, K_\alpha^2 = \frac{eh}{C\alpha} \tag{36}$$

$$K_T = 1 - \frac{K_\vartheta^2}{K_{emgc}^2}, K_\vartheta^2 = \frac{gf}{C\vartheta} \tag{37}$$

$$K_B = 1 - \frac{K_\beta^2}{K_{emgc}^2}, K_\beta^2 = \frac{hg}{C\beta} \tag{38}$$

$$K_Z = 1 - \frac{K_\zeta^2}{K_{emgc}^2}, K_\zeta^2 = \frac{eg}{C\zeta} \tag{39}$$

$$K_S = 1 - \frac{K_\xi^2}{K_{emgc}^2}, K_\xi^2 = \frac{ef}{C\xi} \tag{40}$$

$$K_L = 1 - \frac{K_\lambda^2}{K_{emgc}^2}, K_\lambda^2 = \frac{hf}{C\lambda} \tag{41}$$

3 The *F*-eigenvectors and their properties

The first pair of the eigenvectors called the first and second *F*-eigenvectors is defined below by expressions (42) and (43), respectively. Eigenvector (42) relates to eigenvalues (10) and eigenvector (43) relates to eigenvalues (11). For eigenvector (42), the eigenvector components correspondingly depend on the parameters M_{ef}, M_{hf}, M_{gf}, and M_{ff} that are defined by expressions (28), (30), (31), and (25), respectively, and contain the subscript "*f*" symbolizing the *F*-eigenvectors. So, the *F*-eigenvectors can be written in the following convenient forms:

$$\begin{pmatrix} U^{0(1)} \\ \phi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(2)} \\ \phi^{0(2)} \\ \psi^{0(2)} \\ \Phi^{0(2)} \\ \Psi^{0(2)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \phi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(4)} \\ \phi^{0(4)} \\ \psi^{0(4)} \\ \Phi^{0(4)} \\ \Psi^{0(4)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \phi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(6)} \\ \phi^{0(6)} \\ \psi^{0(6)} \\ \Phi^{0(6)} \\ \Psi^{0(6)} \end{pmatrix} =$$

$$\begin{pmatrix} U^{0(7)} \\ \phi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^{0(8)} \\ \phi^{0(8)} \\ \psi^{0(8)} \\ \Phi^{0(8)} \\ \Psi^{0(8)} \end{pmatrix} = \begin{pmatrix} U_f^{0(7)} = 0 \\ \phi_f^{0(7)} = M_{ef} \\ \psi_f^{0(7)} = M_{hf} \\ \Phi_f^{0(7)} = M_{gf} \\ \Psi_f^{0(7)} = M_{ff} \end{pmatrix} \tag{42}$$

$$\begin{pmatrix} U^{0(9)} \\ \phi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^{0(10)} \\ \phi^{0(10)} \\ \psi^{0(10)} \\ \Phi^{0(10)} \\ \Psi^{0(10)} \end{pmatrix} = \begin{pmatrix} U_f^{0(9)} \\ \phi_f^{0(9)} \\ \psi_f^{0(9)} \\ \Phi_f^{0(9)} \\ \Psi_f^{0(9)} \end{pmatrix} \tag{43}$$

where

$$U_f^{0(9)} = \frac{e\phi_f^{0(9)} + h\psi_f^{0(9)} + g\Phi_f^{0(9)} + f\Psi_f^{0(9)}}{CK_{emgc}^2} = \frac{M_4}{CK_{emgc}^2} \tag{44}$$

$$\phi_f^{0(9)} = \gamma\alpha\lambda K_G K_A K_L + \mu\vartheta\zeta K_M K_T K_Z + \beta^2\xi K_B{}^2 K_S - \alpha\beta\vartheta K_A K_B K_T - \beta\lambda\zeta K_B K_L K_Z - \mu\gamma\xi K_M K_G K_S = \phi_f^{0(7)} - \gamma\alpha\lambda\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) - \mu\vartheta\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \beta^2\xi\left(2\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \alpha\beta\vartheta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) + \beta\lambda\zeta\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \mu\gamma\xi\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) \tag{45}$$

$$\psi_f^{0(9)} = \gamma\alpha\xi K_G K_A K_S + \lambda\zeta^2 K_L K_Z{}^2 + \varepsilon\beta\vartheta K_E K_B K_T - \alpha\vartheta\zeta K_A K_T K_Z - \beta\xi\zeta K_B K_S K_Z - \varepsilon\gamma\lambda K_E K_G K_L = \psi_f^{0(7)} - \gamma\alpha\xi\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) - \lambda\zeta^2\left(\frac{K_\lambda^2}{K_{emgc}^2} + 2\frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\beta\vartheta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) + \alpha\vartheta\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \beta\xi\zeta\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \varepsilon\gamma\lambda\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) \quad (46)$$

$$\Phi_f^{0(9)} = \alpha^2\vartheta K_A{}^2 K_T + \mu\xi\zeta K_M K_S K_Z + \varepsilon\beta\lambda K_E K_B K_L - \alpha\lambda\zeta K_A K_L K_Z - \alpha\beta\xi K_A K_B K_S - \varepsilon\mu\vartheta K_E K_M K_T = \Phi_f^{0(7)} - \alpha^2\vartheta\left(2\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) - \mu\xi\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\beta\lambda\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \alpha\lambda\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \alpha\beta\xi\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \varepsilon\mu\vartheta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) \quad (47)$$

$$\Psi_f^{0(9)} = 2\alpha\beta\zeta K_A K_B K_Z + \varepsilon\mu\gamma K_E K_M K_G - \gamma\alpha^2 K_G K_A{}^2 - \mu\zeta^2 K_M K_Z{}^2 - \varepsilon\beta^2 K_E K_B{}^2 = \Psi_f^{0(7)} - 2\alpha\beta\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\mu\gamma\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_m^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2}\right) + \gamma\alpha^2\left(\frac{K_g^2}{K_{emgc}^2} + 2\frac{K_\alpha^2}{K_{emgc}^2}\right) + \mu\zeta^2\left(\frac{K_m^2}{K_{emgc}^2} + 2\frac{K_\zeta^2}{K_{emgc}^2}\right) + \varepsilon\beta^2\left(\frac{K_e^2}{K_{emgc}^2} + 2\frac{K_\beta^2}{K_{emgc}^2}\right) \quad (48)$$

It is clearly seen in expressions (45), (46), (47), and (48) that the eigenvector components $\phi_f^{0(9)}$, $\psi_f^{0(9)}$, $\Phi_f^{0(9)}$, and $\Psi_f^{0(9)}$ depend on the corresponding eigenvector components $\phi_f^{0(7)}$, $\psi_f^{0(7)}$, $\Phi_f^{0(7)}$, and $\Psi_f^{0(7)}$. This manifests that both eigenvectors (42) and (43) are naturally called the *F*-eigenvectors. The analogical reason was used for distinguishing of the rest three independent

eigenvectors called the *G*-, *H*-, and *E*-eigenvectors in the following three sections.

With the CEMGCMC coupling mechanisms M_4 and M_5 defined by expressions (20) and (21), respectively, the properties of the corresponding components of the *F*-eigenvectors can be demonstrated by all the expressions written below in this section. For the linear combinations of the components of the first *F*-eigenvector, the following five expressions disclose some useful properties:

$$e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)} = e\phi_f^{0(9)} + h\psi_f^{0(9)} + g\Phi_f^{0(9)} + f\Psi_f^{0(9)} = eM_{ef} + hM_{hf} + gM_{gf} + fM_{ff} = M_4 \quad (49)$$

$$\varepsilon\phi_f^{0(7)} + \alpha\psi_f^{0(7)} + \zeta\Phi_f^{0(7)} + \xi\Psi_f^{0(7)} = \varepsilon K_E\phi_f^{0(9)} + \alpha K_A\psi_f^{0(9)} + \zeta K_Z\Phi_f^{0(9)} + \xi K_S\Psi_f^{0(9)} = 0 \quad (50)$$

$$\alpha\phi_f^{0(7)} + \mu\psi_f^{0(7)} + \beta\Phi_f^{0(7)} + \lambda\Psi_f^{0(7)} = \alpha K_A\phi_f^{0(9)} + \mu K_M\psi_f^{0(9)} + \beta K_B\Phi_f^{0(9)} + \lambda K_L\Psi_f^{0(9)} = 0 \quad (51)$$

$$\zeta\phi_f^{0(7)} + \beta\psi_f^{0(7)} + \gamma\Phi_f^{0(7)} + \vartheta\Psi_f^{0(7)} = \zeta K_Z\phi_f^{0(9)} + \beta K_B\psi_f^{0(9)} + \gamma K_G\Phi_f^{0(9)} + \vartheta K_T\Psi_f^{0(9)} = 0 \quad (52)$$

$$\xi\phi_f^{0(7)} + \lambda\psi_f^{0(7)} + \vartheta\Phi_f^{0(7)} + \eta\Psi_f^{0(7)} = \xi M_{ef} + \lambda M_{hf} + \vartheta M_{gf} + \eta M_{ff} = M_5 \quad (53)$$

For the linear combinations of the components of the second *F*-eigenvector, the following equalities can be obtained:

$$\varepsilon\phi_f^{0(9)} + \alpha\psi_f^{0(9)} + \zeta\Phi_f^{0(9)} + \xi\Psi_f^{0(9)} = -\varepsilon K_E\phi_f^{0(7)} - \alpha K_A\psi_f^{0(7)} - \zeta K_Z\Phi_f^{0(7)} - \xi K_S\Psi_f^{0(7)} = \frac{e}{CK_{emgc}^2}\left(e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)}\right) = \frac{eM_4}{CK_{emgc}^2} \quad (54)$$

$$\alpha\phi_f^{0(9)} + \mu\psi_f^{0(9)} + \beta\Phi_f^{0(9)} + \lambda\Psi_f^{0(9)} = -\alpha K_A\phi_f^{0(7)} - \mu K_M\psi_f^{0(7)} - \beta K_B\Phi_f^{0(7)} - \lambda K_L\Psi_f^{0(7)} = \frac{h}{CK_{emgc}^2}\left(e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)}\right) = \frac{hM_4}{CK_{emgc}^2} \quad (55)$$

$$\zeta\phi_f^{0(9)} + \beta\psi_f^{0(9)} + \gamma\Phi_f^{0(9)} + \vartheta\Psi_f^{0(9)} = -\zeta K_Z\phi_f^{0(7)} - \beta K_B\psi_f^{0(7)} - \gamma K_G\Phi_f^{0(7)} - \vartheta K_T\Psi_f^{0(7)} = \frac{g}{CK_{emgc}^2}\left(e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)}\right) = \frac{gM_4}{CK_{emgc}^2} \quad (56)$$

$$\xi K_S\phi_f^{0(9)} + \lambda K_L\psi_f^{0(9)} + \vartheta K_T\Phi_f^{0(9)} + \eta K_F\Psi_f^{0(9)} = 0 \quad (57)$$

Using property (57), it is possible to write down the following equalities:

$$\xi\phi_f^{0(9)} + \lambda\psi_f^{0(9)} + \vartheta\Phi_f^{0(9)} + \eta\Psi_f^{0(9)} = \xi\frac{K_\xi^2}{K_{emgc}^2}\phi_f^{0(9)} + \lambda\frac{K_\lambda^2}{K_{emgc}^2}\psi_f^{0(9)} + \vartheta\frac{K_\vartheta^2}{K_{emgc}^2}\Phi_f^{0(9)} + \eta\frac{K_f^2}{K_{emgc}^2}\Psi_f^{0(9)} = \frac{f}{CK_{emgc}^2}\left(e\phi_f^{0(9)} + h\psi_f^{0(9)} + g\Phi_f^{0(9)} + f\Psi_f^{0(9)}\right) = \frac{f}{CK_{emgc}^2}\left(e\phi_f^{0(7)} + h\psi_f^{0(7)} + g\Phi_f^{0(7)} + f\Psi_f^{0(7)}\right) = \xi\frac{K_\xi^2}{K_{emgc}^2}\phi_f^{0(7)} + \lambda\frac{K_\lambda^2}{K_{emgc}^2}\psi_f^{0(7)} + \vartheta\frac{K_\vartheta^2}{K_{emgc}^2}\Phi_f^{0(7)} + \eta\frac{K_f^2}{K_{emgc}^2}\Psi_f^{0(7)} = \frac{fM_4}{CK_{emgc}^2} \quad (58)$$

4 The *G*-eigenvectors and their properties

The explicit forms of the second pair of the independent eigenvectors called the *G*-eigenvectors are obtained below in expressions (59) and (60). For first *G*-eigenvector (59), the eigenvector components correspondingly depend on the parameters M_{eg}, M_{hg}, M_{gg}, and M_{gf} that are defined by expressions (27), (29), (24), and (31), respectively, and contain the subscript "*g*" symbolizing the *G*-eigenvectors. So, the first and second *G*-eigenvectors respectively read:

$$\begin{pmatrix} U^{0(1)} \\ \phi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(2)} \\ \phi^{0(2)} \\ \psi^{0(2)} \\ \Phi^{0(2)} \\ \Psi^{0(2)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \phi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(4)} \\ \phi^{0(4)} \\ \psi^{0(4)} \\ \Phi^{0(4)} \\ \Psi^{0(4)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \phi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(6)} \\ \phi^{0(6)} \\ \psi^{0(6)} \\ \Phi^{0(6)} \\ \Psi^{0(6)} \end{pmatrix} =$$

$$\begin{pmatrix} U^{0(7)} \\ \phi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^{0(8)} \\ \phi^{0(8)} \\ \psi^{0(8)} \\ \Phi^{0(8)} \\ \Psi^{0(8)} \end{pmatrix} = \begin{pmatrix} U_g^{0(7)} = 0 \\ \phi_g^{0(7)} = M_{eg} \\ \psi_g^{0(7)} = M_{hg} \\ \Phi_g^{0(7)} = M_{gg} \\ \Psi_g^{0(7)} = M_{gf} \end{pmatrix} \tag{59}$$

$$\begin{pmatrix} U^{0(9)} \\ \phi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^{0(10)} \\ \phi^{0(10)} \\ \psi^{0(10)} \\ \Phi^{0(10)} \\ \Psi^{0(10)} \end{pmatrix} = \begin{pmatrix} U_g^{0(9)} \\ \phi_g^{0(9)} \\ \psi_g^{0(9)} \\ \Phi_g^{0(9)} \\ \Psi_g^{0(9)} \end{pmatrix} \tag{60}$$

where

$$U_g^{0(9)} = \frac{e\phi_g^{0(9)} + h\psi_g^{0(9)} + g\Phi_g^{0(9)} + f\Psi_g^{0(9)}}{CK_{emgc}^2} = \frac{M_3}{CK_{emgc}^2} \tag{61}$$

$$\phi_g^{0(9)} = \eta\alpha\beta K_F K_A K_B + \mu\vartheta\xi K_M K_T K_S + \lambda^2\zeta {K_L}^2 K_Z - \alpha\vartheta\lambda K_A K_T K_L - \mu\eta\zeta K_M K_F K_Z - \beta\lambda\xi K_B K_L K_S = \phi_g^{0(7)} - \eta\alpha\beta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2}\right) - \mu\vartheta\xi\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) - \lambda^2\zeta\left(2\frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \alpha\vartheta\lambda\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \mu\eta\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \beta\lambda\xi\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) \tag{62}$$

$$\psi_g^{0(9)} = \eta\alpha\zeta K_F K_A K_Z + \beta\xi^2 K_B {K_S}^2 + \varepsilon\vartheta\lambda K_E K_T K_L - \alpha\vartheta\xi K_A K_T K_S - \lambda\xi\zeta K_L K_S K_Z - \varepsilon\eta\beta K_E K_F K_B = \psi_g^{0(7)} - \eta\alpha\zeta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \beta\xi^2\left(\frac{K_\beta^2}{K_{emgc}^2} + 2\frac{K_\xi^2}{K_{emgc}^2}\right) - \varepsilon\vartheta\lambda\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \alpha\vartheta\xi\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \lambda\xi\zeta\left(\frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \varepsilon\eta\beta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2}\right) \quad (63)$$

$$\Phi_g^{0(9)} = 2\alpha\lambda\xi K_A K_L K_S + \varepsilon\mu\eta K_E K_M K_F - \eta\alpha^2 K_F {K_A}^2 - \mu\xi^2 K_M {K_S}^2 - \varepsilon\lambda^2 K_E {K_L}^2 = \Phi_g^{0(7)} - 2\alpha\lambda\xi\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) - \varepsilon\mu\eta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_m^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2}\right) + \eta\alpha^2\left(\frac{K_f^2}{K_{emgc}^2} + 2\frac{K_\alpha^2}{K_{emgc}^2}\right) + \mu\xi^2\left(\frac{K_m^2}{K_{emgc}^2} + 2\frac{K_\xi^2}{K_{emgc}^2}\right) + \varepsilon\lambda^2\left(\frac{K_e^2}{K_{emgc}^2} + 2\frac{K_\lambda^2}{K_{emgc}^2}\right) \quad (64)$$

$$\Psi_g^{0(9)} = \alpha^2\vartheta {K_A}^2 K_T + \mu\xi\zeta K_M K_S K_Z + \varepsilon\beta\lambda K_E K_B K_L - \alpha\lambda\zeta K_A K_L K_Z - \alpha\beta\xi K_A K_B K_S - \varepsilon\mu\vartheta K_E K_M K_T = \Psi_g^{0(7)} - \alpha^2\vartheta\left(2\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) - \mu\xi\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\beta\lambda\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \alpha\lambda\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \alpha\beta\xi\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \varepsilon\mu\vartheta\left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) \quad (65)$$

The properties of the corresponding *G*-eigenvectors' components can be illuminated by all the equalities written below in this section, where the CEMGCMC coupling mechanism M_3 is defined by expression (19). For the natural linear combinations of the components of the first *G*-eigenvector, the reader can find the following five expressions:

$$e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)} = e\phi_g^{0(9)} + h\psi_g^{0(9)} + g\Phi_g^{0(9)} + f\Psi_g^{0(9)} = eM_{eg} + hM_{hg} + gM_{gg} + fM_{gf} = M_3 \quad (66)$$

$$\varepsilon\phi_g^{0(7)} + \alpha\psi_g^{0(7)} + \zeta\Phi_g^{0(7)} + \xi\Psi_g^{0(7)} = \varepsilon K_E\phi_g^{0(9)} + \alpha K_A\psi_g^{0(9)} + \zeta K_Z\Phi_g^{0(9)} + \xi K_S\Psi_g^{0(9)} = 0 \quad (67)$$

$$\alpha\phi_g^{0(7)} + \mu\psi_g^{0(7)} + \beta\Phi_g^{0(7)} + \lambda\Psi_g^{0(7)} = \alpha K_A\phi_g^{0(9)} + \mu K_M\psi_g^{0(9)} + \beta K_B\Phi_g^{0(9)} + \lambda K_L\Psi_g^{0(9)} = 0 \quad (68)$$

$$\zeta\phi_g^{0(7)} + \beta\psi_g^{0(7)} + \gamma\Phi_g^{0(7)} + \vartheta\Psi_g^{0(7)} = \zeta M_{eg} + \beta M_{hg} + \gamma M_{gg} + \vartheta M_{gf} = M_5 \quad (69)$$

$$\xi\phi_g^{0(7)} + \lambda\psi_g^{0(7)} + \vartheta\Phi_g^{0(7)} + \eta\Psi_g^{0(7)} = \xi K_S\phi_g^{0(9)} + \lambda K_L\psi_g^{0(9)} + \vartheta K_T\Phi_g^{0(9)} + \eta K_F\Psi_g^{0(9)} = 0 \quad (70)$$

For the linear combinations of the components of the second G-eigenvector, it is natural to demonstrate the following obtained properties shown in the five expressions below:

$$\varepsilon\phi_g^{0(9)} + \alpha\psi_g^{0(9)} + \zeta\Phi_g^{0(9)} + \xi\Psi_g^{0(9)} = -\varepsilon K_E\phi_g^{0(7)} - \alpha K_A\psi_g^{0(7)} - \zeta K_Z\Phi_g^{0(7)} - \xi K_S\Psi_g^{0(7)} = \frac{e}{CK_{emgc}^2}\left(e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)}\right) = \frac{eM_3}{CK_{emgc}^2} \quad (71)$$

$$\alpha\phi_g^{0(9)} + \mu\psi_g^{0(9)} + \beta\Phi_g^{0(9)} + \lambda\Psi_g^{0(9)} = -\alpha K_A\phi_g^{0(7)} - \mu K_M\psi_g^{0(7)} - \beta K_B\Phi_g^{0(7)} - \lambda K_L\Psi_g^{0(7)} = \frac{h}{CK_{emgc}^2}\left(e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)}\right) = \frac{hM_3}{CK_{emgc}^2} \quad (72)$$

$$\zeta K_Z\phi_g^{0(9)} + \beta K_B\psi_g^{0(9)} + \gamma K_G\Phi_g^{0(9)} + \vartheta K_T\Psi_g^{0(9)} = 0 \quad (73)$$

Using equality (73), it is possible to obtain the following equalities:

$$\zeta\phi_g^{0(9)} + \beta\psi_g^{0(9)} + \gamma\Phi_g^{0(9)} + \vartheta\Psi_g^{0(9)} = \zeta\frac{K_\zeta^2}{K_{emgc}^2}\phi_g^{0(9)} + \beta\frac{K_\beta^2}{K_{emgc}^2}\psi_g^{0(9)} + \gamma\frac{K_g^2}{K_{emgc}^2}\Phi_g^{0(9)} + \vartheta\frac{K_\vartheta^2}{K_{emgc}^2}\Psi_g^{0(9)} = \frac{g}{CK_{emgc}^2}\left(e\phi_g^{0(9)} + h\psi_g^{0(9)} + g\Phi_g^{0(9)} + \right.$$

$$f\Psi_g^{0(9)}\Big) = \frac{g}{CK_{emgc}^2}\Big(e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)}\Big) = \zeta\frac{K_\zeta^2}{K_{emgc}^2}\phi_g^{0(7)} + \beta\frac{K_\beta^2}{K_{emgc}^2}\psi_g^{0(7)} + \gamma\frac{K_g^2}{K_{emgc}^2}\Phi_g^{0(7)} + \vartheta\frac{K_\vartheta^2}{K_{emgc}^2}\Psi_g^{0(7)} = \frac{gM_3}{CK_{emgc}^2} \quad (74)$$

For the last linear combination, one can get the following equality:

$$\xi\phi_g^{0(9)} + \lambda\psi_g^{0(9)} + \vartheta\Phi_g^{0(9)} + \eta\Psi_g^{0(9)} = -\xi K_S\phi_g^{0(7)} - \lambda K_L\psi_g^{0(7)} - \vartheta K_T\Phi_g^{0(7)} - \eta K_F\Psi_g^{0(7)} = \frac{g}{CK_{emgc}^2}\Big(e\phi_g^{0(7)} + h\psi_g^{0(7)} + g\Phi_g^{0(7)} + f\Psi_g^{0(7)}\Big) = \frac{fM_3}{CK_{emgc}^2} \quad (75)$$

5 The *H*-eigenvectors and their properties

The third independent pair of the suitable eigenvectors is naturally called the *H*-eigenvectors. Their explicit forms are given below in expressions (76) and (77). For first eigenvector (76), the eigenvector components respectively represent the parameters M_{eh}, M_{hh}, M_{hg}, and M_{hf}. These parameters are defined by expressions (26), (23), (29), and (30), respectively, and contain the subscript "*h*" symbolizing the *H*-eigenvectors. Accordingly, the first and second *H*-eigenvectors are respectively written as follows:

$$\begin{pmatrix} U^{0(1)} \\ \phi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(2)} \\ \phi^{0(2)} \\ \psi^{0(2)} \\ \Phi^{0(2)} \\ \Psi^{0(2)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \phi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(4)} \\ \phi^{0(4)} \\ \psi^{0(4)} \\ \Phi^{0(4)} \\ \Psi^{0(4)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \phi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(6)} \\ \phi^{0(6)} \\ \psi^{0(6)} \\ \Phi^{0(6)} \\ \Psi^{0(6)} \end{pmatrix} = \begin{pmatrix} U^{0(7)} \\ \phi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^{0(8)} \\ \phi^{0(8)} \\ \psi^{0(8)} \\ \Phi^{0(8)} \\ \Psi^{0(8)} \end{pmatrix} = \begin{pmatrix} U_h^{0(7)} = 0 \\ \phi_h^{0(7)} = M_{eh} \\ \psi_h^{0(7)} = M_{hh} \\ \Phi_h^{0(7)} = M_{hg} \\ \Psi_h^{0(7)} = M_{hf} \end{pmatrix} \quad (76)$$

$$\begin{pmatrix} U^{0(9)} \\ \phi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^{0(10)} \\ \phi^{0(10)} \\ \psi^{0(10)} \\ \Phi^{0(10)} \\ \Psi^{0(10)} \end{pmatrix} = \begin{pmatrix} U_h^{0(9)} \\ \phi_h^{0(9)} \\ \psi_h^{0(9)} \\ \Phi_h^{0(9)} \\ \Psi_h^{0(9)} \end{pmatrix} \tag{77}$$

where

$$U_h^{0(9)} = \frac{e\phi_h^{0(9)} + h\psi_h^{0(9)} + g\Phi_h^{0(9)} + f\Psi_h^{0(9)}}{CK_{emgc}^2} = \frac{M_2}{CK_{emgc}^2} \tag{78}$$

$$\phi_h^{0(9)} = \alpha\vartheta^2 K_A {K_T}^2 + \eta\beta\zeta K_F K_B K_Z + \gamma\lambda\xi K_G K_L K_S - \gamma\eta\alpha K_G K_F K_A - \vartheta\lambda\zeta K_T K_L K_Z - \beta\vartheta\xi K_B K_T K_S = \phi_h^{0(7)} - \alpha\vartheta^2 \left(\frac{K_\alpha^2}{K_{emgc}^2} + 2\frac{K_\vartheta^2}{K_{emgc}^2}\right) - \eta\beta\zeta \left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \gamma\lambda\xi \left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \gamma\eta\alpha \left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2}\right) + \vartheta\lambda\zeta \left(\frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \beta\vartheta\xi \left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) \tag{79}$$

$$\psi_h^{0(9)} = 2\vartheta\xi\zeta K_T K_S K_Z + \varepsilon\gamma\eta K_E K_G K_F - \gamma\xi^2 K_G {K_S}^2 - \eta\zeta^2 K_F {K_Z}^2 - \varepsilon\vartheta^2 K_E {K_T}^2 = \psi_h^{0(7)} - 2\vartheta\xi\zeta \left(\frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \varepsilon\gamma\eta \left(\frac{K_e^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2}\right) + \gamma\xi^2 \left(\frac{K_g^2}{K_{emgc}^2} + 2\frac{K_\xi^2}{K_{emgc}^2}\right) + \eta\zeta^2 \left(\frac{K_f^2}{K_{emgc}^2} + 2\frac{K_\zeta^2}{K_{emgc}^2}\right) + \varepsilon\vartheta^2 \left(\frac{K_e^2}{K_{emgc}^2} + 2\frac{K_\vartheta^2}{K_{emgc}^2}\right) \tag{80}$$

$$\Phi_h^{0(9)} = \eta\alpha\zeta K_F K_A K_Z + \beta\xi^2 K_B {K_S}^2 + \varepsilon\vartheta\lambda K_E K_T K_L - \lambda\xi\zeta K_L K_S K_Z - \alpha\vartheta\xi K_A K_T K_S - \varepsilon\eta\beta K_E K_F K_B = \Phi_h^{0(7)} - \eta\alpha\zeta \left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) -$$

$$\beta\xi^2\left(\frac{K_\beta^2}{K_{emgc}^2}+2\frac{K_\xi^2}{K_{emgc}^2}\right)-\varepsilon\vartheta\lambda\left(\frac{K_e^2}{K_{emgc}^2}+\frac{K_\vartheta^2}{K_{emgc}^2}+\frac{K_\lambda^2}{K_{emgc}^2}\right)+\lambda\xi\zeta\left(\frac{K_\lambda^2}{K_{emgc}^2}+\frac{K_\xi^2}{K_{emgc}^2}+\frac{K_\zeta^2}{K_{emgc}^2}\right)+\alpha\vartheta\xi\left(\frac{K_\alpha^2}{K_{emgc}^2}+\frac{K_\vartheta^2}{K_{emgc}^2}+\frac{K_\xi^2}{K_{emgc}^2}\right)+\varepsilon\eta\beta\left(\frac{K_e^2}{K_{emgc}^2}+\frac{K_f^2}{K_{emgc}^2}+\frac{K_\beta^2}{K_{emgc}^2}\right) \quad (81)$$

$$\Psi_h^{0(9)}=\gamma\alpha\xi K_G K_A K_S+\varepsilon\beta\vartheta K_E K_B K_T+\lambda\zeta^2 K_L {K_Z}^2-\alpha\vartheta\zeta K_A K_T K_Z-\beta\xi\zeta K_B K_S K_Z-\varepsilon\gamma\lambda K_E K_G K_L=\Psi_h^{0(7)}-\gamma\alpha\xi\left(\frac{K_g^2}{K_{emgc}^2}+\frac{K_\alpha^2}{K_{emgc}^2}+\frac{K_\xi^2}{K_{emgc}^2}\right)-\varepsilon\beta\vartheta\left(\frac{K_e^2}{K_{emgc}^2}+\frac{K_\beta^2}{K_{emgc}^2}+\frac{K_\vartheta^2}{K_{emgc}^2}\right)-\lambda\zeta^2\left(\frac{K_\lambda^2}{K_{emgc}^2}+2\frac{K_\zeta^2}{K_{emgc}^2}\right)+\alpha\vartheta\zeta\left(\frac{K_\alpha^2}{K_{emgc}^2}+\frac{K_\vartheta^2}{K_{emgc}^2}+\frac{K_\zeta^2}{K_{emgc}^2}\right)+\beta\xi\zeta\left(\frac{K_\beta^2}{K_{emgc}^2}+\frac{K_\xi^2}{K_{emgc}^2}+\frac{K_\zeta^2}{K_{emgc}^2}\right)+\varepsilon\gamma\lambda\left(\frac{K_e^2}{K_{emgc}^2}+\frac{K_g^2}{K_{emgc}^2}+\frac{K_\lambda^2}{K_{emgc}^2}\right) \quad (82)$$

The natural linear combinations of the corresponding H-eigenvectors' components disclose the properties obtained in all the expressions written below in this section, where the CEMGCMC coupling mechanisms M_2 and M_5 are defined by equations (18) and (21), respectively. For the linear combinations of the components of the first H-eigenvector, the following five properties can be obtained:

$$e\phi_h^{0(7)}+h\psi_h^{0(7)}+g\Phi_h^{0(7)}+f\Psi_h^{0(7)}=e\phi_h^{0(9)}+h\psi_h^{0(9)}+g\Phi_h^{0(9)}+f\Psi_h^{0(9)}=eM_{eh}+hM_{hh}+gM_{hg}+fM_{hf}=M_2 \quad (83)$$

$$\varepsilon\phi_h^{0(7)}+\alpha\psi_h^{0(7)}+\zeta\Phi_h^{0(7)}+\xi\Psi_h^{0(7)}=\varepsilon K_E\phi_h^{0(9)}+\alpha K_A\psi_h^{0(9)}+\zeta K_Z\Phi_h^{0(9)}+\xi K_S\Psi_h^{0(9)}=0 \quad (84)$$

$$\alpha\phi_h^{0(7)}+\mu\psi_h^{0(7)}+\beta\Phi_h^{0(7)}+\lambda\Psi_h^{0(7)}=\alpha M_{eh}+\mu M_{hh}+\beta M_{hg}+\lambda M_{hf}=M_5 \quad (85)$$

$$\zeta\phi_h^{0(7)}+\beta\psi_h^{0(7)}+\gamma\Phi_h^{0(7)}+\vartheta\Psi_h^{0(7)}=\zeta K_Z\phi_h^{0(9)}+\beta K_B\psi_h^{0(9)}+\gamma K_G\Phi_h^{0(9)}+\vartheta K_T\Psi_h^{0(9)}=0 \quad (86)$$

$$\xi\phi_h^{0(7)}+\lambda\psi_h^{0(7)}+\vartheta\Phi_h^{0(7)}+\eta\Psi_h^{0(7)}=\xi K_S\phi_h^{0(9)}+\lambda K_L\psi_h^{0(9)}+\vartheta K_T\Phi_h^{0(9)}+\eta K_F\Psi_h^{0(9)}=0 \quad (87)$$

For the linear combinations of the components of the second H-eigenvector there are also the five properties written down in the following last five expressions for this section:

$$\varepsilon\phi_h^{0(9)} + \alpha\psi_h^{0(9)} + \zeta\Phi_h^{0(9)} + \xi\Psi_h^{0(9)} = -\varepsilon K_E\phi_h^{0(7)} - \alpha K_A\psi_h^{0(7)} - \zeta K_Z\Phi_h^{0(7)} - \xi K_S\Psi_h^{0(7)} = \frac{e}{CK_{emgc}^2}\left(e\phi_h^{0(7)} + h\psi_h^{0(7)} + g\Phi_h^{0(7)} + f\Psi_h^{0(7)}\right) = \frac{eM_2}{CK_{emgc}^2} \quad (88)$$

$$\alpha K_A\phi_h^{0(9)} + \mu K_M\psi_h^{0(9)} + \beta K_B\Phi_h^{0(9)} + \lambda K_L\Psi_h^{0(9)} = 0 \quad (89)$$

Utilizing expression (89), the following equalities in expression (90) can be obtained:

$$\alpha\phi_h^{0(9)} + \mu\psi_h^{0(9)} + \beta\Phi_h^{0(9)} + \lambda\Psi_h^{0(9)} = \alpha\frac{K_\alpha^2}{K_{emgc}^2}\phi_h^{0(9)} + \mu\frac{K_m^2}{K_{emgc}^2}\psi_h^{0(9)} + \beta\frac{K_\beta^2}{K_{emgc}^2}\Phi_h^{0(9)} + \lambda\frac{K_\lambda^2}{K_{emgc}^2}\Psi_h^{0(9)} = \frac{h}{CK_{emgc}^2}\left(e\phi_h^{0(9)} + h\psi_h^{0(9)} + g\Phi_h^{0(9)} + f\Psi_h^{0(9)}\right) = \frac{h}{CK_{emgc}^2}\left(e\phi_h^{0(7)} + h\psi_h^{0(7)} + g\Phi_h^{0(7)} + f\Psi_h^{0(7)}\right) = \alpha\frac{K_\alpha^2}{K_{emgc}^2}\phi_h^{0(7)} + \mu\frac{K_m^2}{K_{emgc}^2}\psi_h^{0(7)} + \beta\frac{K_\beta^2}{K_{emgc}^2}\Phi_h^{0(7)} + \lambda\frac{K_\lambda^2}{K_{emgc}^2}\Psi_h^{0(7)} = \frac{hM_2}{CK_{emgc}^2} \quad (90)$$

For this section, the last two natural linear combinations demonstrate the following properties:

$$\zeta\phi_h^{0(9)} + \beta\psi_h^{0(9)} + \gamma\Phi_h^{0(9)} + \vartheta\Psi_h^{0(9)} = -\zeta K_Z\phi_h^{0(7)} - \beta K_B\psi_h^{0(7)} - \gamma K_G\Phi_h^{0(7)} - \vartheta K_T\Psi_h^{0(7)} = \frac{g}{CK_{emgc}^2}\left(e\phi_h^{0(7)} + h\psi_h^{0(7)} + g\Phi_h^{0(7)} + f\Psi_h^{0(7)}\right) = \frac{gM_2}{CK_{emgc}^2} \quad (91)$$

$$\xi\phi_h^{0(9)} + \lambda\psi_h^{0(9)} + \vartheta\Phi_h^{0(9)} + \eta\Psi_h^{0(9)} = -\xi K_S\phi_h^{0(7)} - \lambda K_L\psi_h^{0(7)} - \vartheta K_T\Phi_h^{0(7)} - \eta K_F\Psi_h^{0(7)} = \frac{f}{CK_{emgc}^2}\left(e\phi_h^{0(7)} + h\psi_h^{0(7)} + g\Phi_h^{0(7)} + f\Psi_h^{0(7)}\right) = \frac{fM_2}{CK_{emgc}^2} \quad (92)$$

6 The *E*-eigenvectors and their properties

The last independent pair of the eigenvectors is called the *E*-eigenvectors. Expressions (93) and (94) written below stand for the first and second *E*-eigenvectors, respectively. The corresponding components of the first *E*-eigenvector are naturally equal to parameters M_{ee} (22), M_{eh} (26), M_{eg} (27), and M_{ef} (28). Here, subscript "*e*" symbolizes the *E*-eigenvectors. Therefore, the *E*-eigenvectors are

$$\begin{pmatrix} U^{0(1)} \\ \phi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(2)} \\ \phi^{0(2)} \\ \psi^{0(2)} \\ \Phi^{0(2)} \\ \Psi^{0(2)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \phi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(4)} \\ \phi^{0(4)} \\ \psi^{0(4)} \\ \Phi^{0(4)} \\ \Psi^{0(4)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \phi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(6)} \\ \phi^{0(6)} \\ \psi^{0(6)} \\ \Phi^{0(6)} \\ \Psi^{0(6)} \end{pmatrix} =$$

$$\begin{pmatrix} U^{0(7)} \\ \phi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^{0(8)} \\ \phi^{0(8)} \\ \psi^{0(8)} \\ \Phi^{0(8)} \\ \Psi^{0(8)} \end{pmatrix} = \begin{pmatrix} U_e^{0(7)} = 0 \\ \phi_e^{0(7)} = M_{ee} \\ \psi_e^{0(7)} = M_{eh} \\ \Phi_e^{0(7)} = M_{eg} \\ \Psi_e^{0(7)} = M_{ef} \end{pmatrix} \tag{93}$$

$$\begin{pmatrix} U^{0(9)} \\ \phi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^{0(10)} \\ \phi^{0(10)} \\ \psi^{0(10)} \\ \Phi^{0(10)} \\ \Psi^{0(10)} \end{pmatrix} = \begin{pmatrix} U_e^{0(9)} \\ \phi_e^{0(9)} \\ \psi_e^{0(9)} \\ \Phi_e^{0(9)} \\ \Psi_e^{0(9)} \end{pmatrix} \tag{94}$$

where

$$U_e^{0(9)} = \frac{e\phi_e^{0(9)} + h\psi_e^{0(9)} + g\Phi_e^{0(9)} + f\Psi_e^{0(9)}}{CK_{emgc}^2} = \frac{M_1}{CK_{emgc}^2} \tag{95}$$

$$\phi_e^{0(9)} = 2\beta\vartheta\lambda K_B K_T K_L + \mu\gamma\eta K_M K_G K_F - \gamma\lambda^2 K_G {K_L}^2 - \eta\beta^2 K_F {K_B}^2 - \mu\vartheta^2 K_M {K_T}^2 = \phi_e^{0(7)} - 2\beta\vartheta\lambda\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) - \mu\gamma\eta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2}\right) + \gamma\lambda^2\left(\frac{K_g^2}{K_{emgc}^2} + 2\frac{K_\lambda^2}{K_{emgc}^2}\right) + \eta\beta^2\left(\frac{K_f^2}{K_{emgc}^2} + 2\frac{K_\beta^2}{K_{emgc}^2}\right) + \mu\vartheta^2\left(\frac{K_m^2}{K_{emgc}^2} + 2\frac{K_\vartheta^2}{K_{emgc}^2}\right) \quad (96)$$

$$\psi_e^{0(9)} = \alpha\vartheta^2 K_A {K_T}^2 + \eta\beta\zeta K_F K_B K_Z + \gamma\lambda\xi K_G K_L K_S - \gamma\eta\alpha K_G K_F K_A - \beta\vartheta\xi K_B K_T K_S - \vartheta\lambda\zeta K_T K_L K_Z = \psi_e^{0(7)} - \alpha\vartheta^2\left(\frac{K_\alpha^2}{K_{emgc}^2} + 2\frac{K_\vartheta^2}{K_{emgc}^2}\right) - \eta\beta\zeta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \gamma\lambda\xi\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \gamma\eta\alpha\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2}\right) + \beta\vartheta\xi\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \vartheta\lambda\zeta\left(\frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) \quad (97)$$

$$\Phi_e^{0(9)} = \eta\alpha\beta K_F K_A K_B + \lambda^2\zeta {K_L}^2 K_Z + \mu\vartheta\xi K_M K_T K_S - \beta\lambda\xi K_B K_L K_S - \alpha\vartheta\lambda K_A K_T K_L - \mu\eta\zeta K_M K_F K_Z = \Phi_e^{0(7)} - \eta\alpha\beta\left(\frac{K_f^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2}\right) - \lambda^2\zeta\left(2\frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \mu\vartheta\xi\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \beta\lambda\xi\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \alpha\vartheta\lambda\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) + \mu\eta\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_f^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) \quad (98)$$

$$\Psi_e^{0(9)} = \gamma\alpha\lambda K_G K_A K_L + \mu\vartheta\zeta K_M K_T K_Z + \beta^2\xi {K_B}^2 K_S - \alpha\beta\vartheta K_A K_B K_T - \beta\lambda\zeta K_B K_L K_Z - \mu\gamma\xi K_M K_G K_S = \Psi_e^{0(7)} - \gamma\alpha\lambda\left(\frac{K_g^2}{K_{emgc}^2} + \frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2}\right) - \mu\vartheta\zeta\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) - \beta^2\xi\left(2\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) + \alpha\beta\vartheta\left(\frac{K_\alpha^2}{K_{emgc}^2} + \frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\vartheta^2}{K_{emgc}^2}\right) + \beta\lambda\zeta\left(\frac{K_\beta^2}{K_{emgc}^2} + \frac{K_\lambda^2}{K_{emgc}^2} + \frac{K_\zeta^2}{K_{emgc}^2}\right) + \mu\gamma\xi\left(\frac{K_m^2}{K_{emgc}^2} + \frac{K_g^2}{K_{emgc}^2} + \frac{K_\xi^2}{K_{emgc}^2}\right) \quad (99)$$

With the natural linear combinations of the suitable *E*-eigenvectors' components, this pair of the *E*-eigenvectors possesses the properties demonstrated in the rest ten expressions written below, where M_1 in expression

(100) stands for CEMGCMC coupling mechanism (17). The following first five expressions relate to the linear combinations of the components of the first E-eigenvector:

$$e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)} = e\phi_e^{0(9)} + h\psi_e^{0(9)} + g\Phi_e^{0(9)} + f\Psi_e^{0(9)} = eM_{ee} + hM_{eh} + gM_{eg} + fM_{ef} = M_1 \quad (100)$$

$$\varepsilon\phi_e^{0(7)} + \alpha\psi_e^{0(7)} + \zeta\Phi_e^{0(7)} + \xi\Psi_e^{0(7)} = \varepsilon M_{ee} + \alpha M_{eh} + \zeta M_{eg} + \xi M_{ef} = M_5 \quad (101)$$

$$\alpha\phi_e^{0(7)} + \mu\psi_e^{0(7)} + \beta\Phi_e^{0(7)} + \lambda\Psi_e^{0(7)} = \alpha K_A\phi_e^{0(9)} + \mu K_M\psi_e^{0(9)} + \beta K_B\Phi_e^{0(9)} + \lambda K_L\Psi_e^{0(9)} = 0 \quad (102)$$

$$\zeta\phi_e^{0(7)} + \beta\psi_e^{0(7)} + \gamma\Phi_e^{0(7)} + \vartheta\Psi_e^{0(7)} = \zeta K_Z\phi_e^{0(9)} + \beta K_B\psi_e^{0(9)} + \gamma K_G\Phi_e^{0(9)} + \vartheta K_T\Psi_e^{0(9)} = 0 \quad (103)$$

$$\xi\phi_e^{0(7)} + \lambda\psi_e^{0(7)} + \vartheta\Phi_e^{0(7)} + \eta\Psi_e^{0(7)} = \xi K_S\phi_e^{0(9)} + \lambda K_L\psi_e^{0(9)} + \vartheta K_T\Phi_e^{0(9)} + \eta K_F\Psi_e^{0(9)} = 0 \quad (104)$$

Concerning the properties of the second E-eigenvector, the following equality is useful:

$$\varepsilon K_E\phi_e^{0(9)} + \alpha K_A\psi_e^{0(9)} + \zeta K_Z\Phi_e^{0(9)} + \xi K_S\Psi_e^{0(9)} = 0 \quad (105)$$

With expressions (105), (32), (36), (39), and (40), the following property can be obtained:

$$\varepsilon\phi_e^{0(9)} + \alpha\psi_e^{0(9)} + \zeta\Phi_e^{0(9)} + \xi\Psi_e^{0(9)} = \varepsilon\frac{K_e^2}{K_{emgc}^2}\phi_e^{0(9)} + \alpha\frac{K_\alpha^2}{K_{emgc}^2}\psi_e^{0(9)} + \zeta\frac{K_\zeta^2}{K_{emgc}^2}\Phi_e^{0(9)} + \xi\frac{K_\xi^2}{K_{emgc}^2}\Psi_e^{0(9)} = \frac{e}{CK_{emgc}^2}\left(e\phi_e^{0(9)} + h\psi_e^{0(9)} + g\Phi_e^{0(9)} + \right.$$

$$f\Psi_e^{0(9)}\Big) = \frac{e}{CK_{emgc}^2}\Big(e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)}\Big) = \varepsilon\frac{K_e^2}{K_{emgc}^2}\phi_e^{0(7)} + \alpha\frac{K_\alpha^2}{K_{emgc}^2}\psi_e^{0(7)} + \zeta\frac{K_\zeta^2}{K_{emgc}^2}\Phi_e^{0(7)} + \xi\frac{K_\xi^2}{K_{emgc}^2}\Psi_e^{0(7)} = \frac{eM_1}{CK_{emgc}^2} \quad (106)$$

The final three expressions for the linear combinations of the components of the second E-eigenvector are

$$\alpha\phi_e^{0(9)} + \mu\psi_e^{0(9)} + \beta\Phi_e^{0(9)} + \lambda\Psi_e^{0(9)} = -\alpha K_A\phi_e^{0(7)} - \mu K_M\psi_e^{0(7)} - \beta K_B\Phi_e^{0(7)} - \lambda K_L\Psi_e^{0(7)} = \frac{h}{CK_{emgc}^2}\Big(e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)}\Big) = \frac{hM_1}{CK_{emgc}^2} \quad (107)$$

$$\zeta\phi_e^{0(9)} + \beta\psi_e^{0(9)} + \gamma\Phi_e^{0(9)} + \vartheta\Psi_e^{0(9)} = -\zeta K_Z\phi_e^{0(7)} - \beta K_B\psi_e^{0(7)} - \gamma K_G\Phi_e^{0(7)} - \vartheta K_T\Psi_e^{0(7)} = \frac{g}{CK_{emgc}^2}\Big(e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)}\Big) = \frac{gM_1}{CK_{emgc}^2} \quad (108)$$

$$\xi\phi_e^{0(9)} + \lambda\psi_e^{0(9)} + \vartheta\Phi_e^{0(9)} + \eta\Psi_e^{0(9)} = -\xi K_S\phi_e^{0(7)} - \lambda K_L\psi_e^{0(7)} - \vartheta K_T\Phi_e^{0(7)} - \eta K_F\Psi_e^{0(7)} = \frac{f}{CK_{emgc}^2}\Big(e\phi_e^{0(7)} + h\psi_e^{0(7)} + g\Phi_e^{0(7)} + f\Psi_e^{0(7)}\Big) = \frac{fM_1}{CK_{emgc}^2} \quad (109)$$

So, this paper has obtained the four pairs of the main independent eigenvectors called the F-eigenvectors (third section), G-eigenvectors (fourth section), H-eigenvectors (fifth section), and E-eigenvectors (this sixth section). Their properties are described in the corresponding section below each pair of the corresponding eigenvectors. All the four pairs of the main independent eigenvectors are written above in convenient explicit forms. However, the reader can find that their forms are quite complicated and to verify their properties is not easy to perform. Indeed, it was not easy to find that all the twenty-four possible eigenvectors (paper [24] has obtained only half of them) reduce to one of the four pairs of the main independent eigenvectors and then to demonstrate the properties of the F-, G-, H-, and E-eigenvectors.

All the properties obtained in this and the previous three sections are the main results of these theoretical investigations. In this section, property (100) demonstrates that the E-eigenvectors relate to the coupling mechanism M_1 (17) of the CEMGCMC K_{emgc}^2 (16). Corresponding properties (49), (66), and (83) in

the previous three sections manifest that the *F*-, *G*-, and *H*-eigenvectors relate to the CEMGCMC coupling mechanisms M_4, M_3, and M_2 defined by formulae (20), (19), and (18), respectively. Moreover, all of these four independent eigenvectors also relate to the fifth coupling mechanisms M_5 (21). Properties (53), (69), (85), and (101) support this statement. It is necessary to state that each of these five coupling mechanisms can release propagation velocities of the 4P-SH acoustic waves guided by the surface of the solid, the interface between two dissimilar solids, or plate (thin film) waveguide. All the properties of the four eigenvectors obtained in this and the previous three sections can be actually useful for analysis of various boundary conditions' determinants to obtain suitable propagation velocities. Indeed, suitable properties can be used for representing final results of propagation speeds in compact and convenient forms because many linear combinations of the eigenvector components (i.e. properties) are equal to zero. This will be useful even in numerical calculations because it is obvious that many possible final results of propagation speeds will be very complicated due to the existence of a lot of material parameters listed in table 1. Also, for the calculations it is preferable to use measured values of the parameters listed in the table. Note that measured values of the material parameters listed in the last several rows of table 1 are currently absent because their measurements require creation of perfect experimental tools.

It is necessary to mention that the acoustic waves are very slow in comparison with the speed of light, C_L, and can propagate in solids at the speeds of $\sim 10^3$ m/s. For the mechanical subsystem, the treatment of additional subsystems such as the electrical, magnetic, gravitational, and cogravitational can result in the fact that the propagation speeds of the acoustic waves can be slightly higher. However, the presence of these additional subsystems allows one to deal with smart materials because any change in one subsystem can result in some response changes in the other subsystems. This fact allows the constitution of various smart technical devices.

7 Discussion

It is necessary to mention also that this work belongs to the case of investigations of the piezogravitocogravitoelectromagnetic materials [1, 31] possessing the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems. Namely this paper further develops the results obtained in pioneer work [1] and paper [24]. It is now possible to compare the

obtained results in this work with the results obtained in book [12] for the case of acoustic wave propagation in piezoelectromagnetics (PEMs) possessing only the mechanical, electrical, and magnetic subsystems, i.e. without taking into account the gravitational phenomena. For the case of the PEM materials there are only two possible pairs of the main independent eigenvectors and they are quite simple compared with the case described in this paper above. However, the investigations of the PEM materials also have peculiarities described in papers [32] and [33]. To ignore these peculiarities can result in obtaining incorrect final solution for the acoustic wave propagation velocities in the PEM materials. The reader can also find single review [14] on the subject of the acoustic wave propagation in the PEM materials. The common problem representing the peculiarity of this study and study [12] is that any eigenvector component does not depend on the propagation velocity. It is worth noting that the cubic PEM materials do not possess such peculiarity because the eigenvectors depend on the propagation velocity in the cubic case treated in book [13]. However, the cubic PEM materials have the other peculiarities. Concerning some investigations of the piezogravitocogravitoelectromagnetic materials with the cubic symmetry, the development of theoretical backgrounds is also possible in the future.

It is now possible to briefly discuss the most exciting thing for the reader concerning the possibility of the instant interplanetary (tele)communication. It was mentioned in the introduction that in the solids, the slow speed of the new 4P-SH-SAW [1] can naturally depend on the speed of the electromagnetic wave, speed of the gravitational wave, and the speeds $\Lambda_{01} = (\zeta_0\lambda_0)^{-1/2} \rightarrow 10^{13}C_L$ and $\Lambda_{02} = (\xi_0\beta_0)^{-1/2} \rightarrow 10^{13}C_L$ of the new fast waves that can be thirteen orders faster than the speed of light, C_L. Also, it is necessary to state that any acoustic wave cannot propagate in a vacuum. So, the transition of information in a vacuum is only possible with one of the following speeds: C_L, Λ_{01}, and Λ_{02}. It is obvious that the speed C_L is too slow for the instant interplanetary communication. On the other hand, the speeds Λ_{01}, and Λ_{02} that thirteen orders faster than C_L are already enough for the instant interplanetary (and even intergalactic) communication. Therefore, it is necessary to develop technologies and infrastructure in order to have the instant interplanetary communication. Indeed, it is not easy and fast. First of all, it is necessary to measure the vacuum parameters ζ_0, λ_0, ξ_0, and β_0 called the gravitoelectric, cogravitomagnetic, cogravitoelectric, and gravitomagnetic constants, respectively. This will allow the human civilization

to know the complete set of the vacuum parameters and the exact values of the speeds Λ_{01} and Λ_{02}.

The C_L-communication due to the well-developed C_L-infrastructure is well-developed on the Earth. This successful development lasted during the last several centuries and involved millions of researchers and engineers on this planet. This means that trillions of the United States dollars were accumulated to have the success in the development of the C_L-communication that is now used even for the slow interplanetary communication. The C_L-communication simultaneously incorporates only two subsystems (electrical and magnetic) to transfer signals through a vacuum. For the mobile C_L-communication, many kinds of mobile devices are commercially available today, for instance, mobile phones even for children. For the Λ_{01}- and Λ_{02}-communication (say Λ-communication below) it is possible to use some interplanetary infrastructure schematically shown and discussed in paper [10]. It is assumed that the development of some mobile Λ-communication can be the final aim after the successful development of the voice and video Λ-communications. For this purpose, it is useful to know the complete set of the material constants for solids listed in table 1.

There are millions of different solids (monocrystals, alloys, composites, etc.) and their wave characteristics can be very important to constitute various (smart) technical devices. However, it is necessary to state that the contributions from both the piezoelectric and piezomagnetic effects to the speed of the acoustic wave are in general small but well-measurable today. Even the contribution from the magnetoelectric effect for the smart PEM materials is well-measurable today in spite of its weakness. The magnetoelectric effect is characterized by the electromagnetic constant α listed in table 1. This constant symbolizes some exchange between the electrical and magnetic subsystems with the following evaluated exchange speed: $V_\alpha = 1/\alpha$. The exchange speed V_α must be faster than the speed of the electromagnetic wave V_{EM} in a PEM solid, $V_\alpha > V_{EM} = (\varepsilon\mu)^{-1/2}$ because the following condition of thermodynamic stability $\alpha^2 < \varepsilon\mu$ [34, 35] must hold. Practically there is always $\alpha^2 << \varepsilon\mu$ ($V_\alpha >> V_{EM}$) for the magnetoelectric solids. Even for strong magnetoelectric composites the speed V_α is several orders faster than the speed V_{EM}. For majority of the magnetoelectric solids, it is assumed that the difference between the speeds V_α and V_{EM} can even reach ten orders. Indeed, this is a huge difference. The thermodynamic stability condition $V_\alpha >> V_{EM}$ for the system representing the electromagnetic wave in a solid or a

vacuum means that the exchange speed V_α between the electrical and magnetic subsystems should be significantly higher the speed V_{EM} of the whole system. This allows the whole system to thermodynamically exist and propagate for long distances in continuous media such as a vacuum and solids.

For the gravitational and cogravitational subsystems there is the gravitocogravitic constant ϑ listed in table 1 and the evaluated exchange speed is $V_\vartheta = 1/\vartheta$. The corresponding thermodynamic stability condition can be written as follows: $\vartheta^2 < \gamma\eta$ and $V_\vartheta > V_{GC} = (\gamma\eta)^{-1/2}$ (even $\vartheta^2 << \gamma\eta$ and $V_\vartheta >> V_{GC}$) where the gravitic constant γ and the cogravitic constant η are also listed in table 1. It is natural to expect that minimum values for the exchange speeds V_α and V_ϑ can be comparable. Some exchange possibilities must exist among the gravitational, cogravitational, electrical, and magnetic subsystems. The possible exchange processes between each pair of the four subsystems can be characterized by the material constants ζ, λ, ξ, and β listed in the last four rows of table 1. These constants introduce the following exchange speeds: $\Lambda_1 = (\zeta\lambda)^{-1/2} \rightarrow 10^{13}C_L$ and $\Lambda_2 = (\xi\beta)^{-1/2} \rightarrow 10^{13}C_L$. The thermodynamic stability conditions can be written here as follows: $\Lambda_1 > V_\alpha > V_{EM}$, $\Lambda_2 > V_\alpha > V_{EM}$, $\Lambda_1 > V_\vartheta > V_{GC}$, $\Lambda_2 > V_\vartheta > V_{GC}$ or even $\Lambda_1 >> V_\alpha >> V_{EM}$, $\Lambda_2 >> V_\alpha >> V_{EM}$, $\Lambda_1 >> V_\vartheta >> V_{GC}$, $\Lambda_2 >> V_\vartheta >> V_{GC}$. The contributions from all of these speeds to the speed of the acoustic wave defined by formula (12) are complicatedly hybridized in the CEMGCMC K^2_{emgc} (13) that possesses the five different coupling mechanisms in form (16).

For a vacuum, it is also possible to assume that there are the following thermodynamic stability conditions: $\Lambda_{01} > V_{\alpha 0} > C_L$, $\Lambda_{02} > V_{\alpha 0} > C_L$, $\Lambda_{01} > V_{\vartheta 0} > C_L$, $\Lambda_{02} > V_{\vartheta 0} > C_L$ or even $\Lambda_{01} >> V_{\alpha 0} >> C_L$, $\Lambda_{02} >> V_{\alpha 0} >> C_L$, $\Lambda_{01} >> V_{\vartheta 0} >> C_L$, $\Lambda_{02} >> V_{\vartheta 0} >> C_L$, where $V_{\alpha 0} = 1/\alpha_0$ and $V_{\vartheta 0} = 1/\vartheta_0$. The most popular case of interactions among the gravitational, cogravitational, electrical, and magnetic subsystems in a vacuum is the captivation of an electromagnetic wave propagating close enough to a gigantic black hole. This case was also discussed in paper [31]. In this case, the black hole gravitation can change the propagation direction of the electromagnetic wave and the wave will be finally directed towards the black hole. It is obvious that these interactions must be significantly faster than the speed of light, C_L, i.e. at the fast speeds Λ_{01} and Λ_{02}. This allows one to treat the system (the electromagnetic wave) as a stationary system at an equilibrium. Each interaction is negligibly small that allows the system to relax after each exchange among the gravitational, cogravitational, electrical, and magnetic fields. However, an enormous number of these interactions can result

in significant change in the propagation direction of the electromagnetic wave while the propagation speed C_L can stay the same because a vacuum represents an isotropic continuum, i.e. the value of the speed of light is the same in any propagation direction. Also, it is assumed that some change in energy of the electromagnetic wave can exist.

8 Conclusion

This theoretical research has obtained the *E*-, *H*-, *G*-, and *F*-eigenvectors. They are the four pairs of the main independent eigenvectors. The properties of the obtained eigenvectors were also demonstrated. It was found that some suitable linear combinations of the eigenvector components can be equal to one of the CEMGCMC coupling mechanisms M_1, M_2, M_3, M_4, and M_5 defined by expressions (17), (18), (19), (20), and (21), respectively. These eigenvectors (namely the CEMGCMC coupling mechanisms) can define the propagation velocities in the theory of the acoustic wave propagation in the transversely isotropic materials of the symmetry class 6 *mm*. The acoustic waves propagate in the directions when there is some coupling among the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems. It is expected that the coupling with some gravitational phenomena will allow the human civilization to develop gravitoelectromagnetic technologies for instant interplanetary (interstellar, intergalactic) communications. Indeed, the instant intergalactic Internet is necessary that can be based on the new fast four-potential gravitocogravitoelectromagnetic waves instead of the conventional two-potential electromagnetic waves propagating at the speed of light. The cooperative development of the electromagnetic and gravitational technologies can be feasible already to the end of this century because the four-potential waves can propagate in a vacuum at the speed thirteen orders faster than the speed of light.

References

[1] A.A. Zakharenko, On piezogravitocogravitoelectromagnetic shear-horizontal acoustic waves. Canadian Journal of Pure and Applied Sciences, 10 (3) (2016), 4011-4028; DOI: https://doi.org/10.5281/zenodo.1301184.

[2] A. Einstein, Die Grundlage der allgemeinen Relativitätstheorie. Annalen der Physik, 354 (7) (1916), 769-822. Version of Record online: 14 MAR 2006, DOI: https://doi.org/10.1002/andp.19163540702.

[3] B.P. Abbott, R. Abbott, T.D. Abbott, M.R. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R.X. Adhikari *et al.*, Observation of gravitational waves from a binary black hole merger. Physical Review Letters, 116 (6) (2016), 061102, 16 pages.

[4] B.M. Yavorsky, A.A. Detlaf and A.K. Lebedev, *The Physics Reference Book for Engineers and Student of the Higher Education*, 8th edition, ONICS Publishers, Moscow (2006), 1054 pages.

[5] O. Heaviside, A gravitational and electromagnetic analogy. The Electrician, 31 (Part I) (1893), 281-282 and 359.

[6] O.D. Jefimenko, *Gravitation and Cogravitation. Developing Newton's Theory of Gravitation to its Physical and Mathematical Conclusion*, Electret Scientific Publishing, USA (2006), 367 pages.

[7] N.A. Kozyrev, *Collected papers*, Leningrad University Publishing, Leningrad, USSR (1991), 448 pages; ISBN 5-288-00626-1, in Russian.

[8] P.S. Laplace, *Exposition du Systeme du Monde*, the fifth book, Nauka Publishers, Leningrad, USSR (1982), 309 pages, in Russian; the English translation by J. Pound, *The System of the World*, two volumes, Kessinger (2007) ISBN 1432653679 [J. Pound, *The System of the World*, two volumes, Richard Phillips, London, UK (1809)].

[9] G.L. Le Sage, *Essai de Chymie Mechanique*, (1761), 121 pages.

[10] A.A. Zakharenko, On necessity of development of instant interplanetary telecommunication based on some gravitational phenomena for remote medical diagnostics and treatment. Canadian Journal of Pure and Applied Sciences, 12 (2) (2018), 4481-4487; DOI: https://doi.org/10.5281/zenodo.1301289.

[11] A.A. Zakharenko, Acoustic wave propagation in (layered) magnetoelectroelastic (composite) materials and influence of gravitational subsystems. The 1st International Symposium on Mechanics, Aberdeen, Scotland, UK, 9-12 July 2018, Jones session: Mechanics of composite materials, DOI: https://doi.org/10.5281/zenodo.1307174. It is available online at https://mechanics.nscj.co.uk/sessions/Jones.html.

[12] A.A. Zakharenko, *Propagation of Seven New SH-SAWs in Piezoelectromagnetics of Class 6 mm*, LAP LAMBERT Academic Publishing GmbH & Co. KG, Saarbruecken-Krasnoyarsk, Germany (2010), 84 pages, ISBN: 978-3-8433-6403-4. DOI: https://doi.org/10.13140/2.1.4000.8645.

[13] A.A. Zakharenko, *Seven New SH-SAWs in Cubic Piezoelectromagnetics*, LAP LAMBERT Academic Publishing GmbH & Co. KG, Saarbruecken-

Krasnoyarsk, Germany (2011), 172 pages, ISBN: 978-3-8473-3485-9. DOI: https://doi.org/10.13140/2.1.2428.0001.

[14] A.A. Zakharenko, Piezoelectromagnetic SH-SAWs: A review. Canadian Journal of Pure and Applied Sciences, 7 (1) (2013), 2227-2240; DOI: https://doi.org/10.5281/zenodo.1300699.

[15] K. Durdag, Wireless surface acoustic wave sensors. Sensors and Transducers Journal, 106 (7) (2009), 1-5.

[16] A. Nechibvute, A. Chawanda, and P. Luhanga, Piezoelectric energy harvesting devices: An alternative energy source for wireless sensors. Smart Materials Research (Hindawi Publishing Corporation), Volume 2012 (2012), Article ID 853481, 13 pages, DOI: http://dx.doi.org/10.1155/2012/853481.

[17] Sh. Priya, H.-Ch. Song, Yu. Zhou, R. Varghese, A. Chopra, S.-G. Kim, I. Kanno, L. Wu, D.S. Ha, J. Ryu, R.G. Polcawich, A review on piezoelectric energy harvesting: Materials, methods, and circuits. Energy Harvesting and Systems, 4 (1) (2017), 3-39, DOI: http://doi.org/10.1515/ehs-2016-0028.

[18] P.K. Sharma and P.V. Baredar, Analysis on piezoelectric energy harvesting small scale device – a review. Journal of King Saud University - Science (Production and hosting by Elsevier B.V.), 31 (4) (2019), 9 pages, in press, DOI: https://doi.org/10.1016/j.jksus.2017.11.002.

[19] J. Iannacci, Microsystem based Energy Harvesting (EH-MEMS): Powering pervasivity of the Internet of Things (IoT) – A review with focus on mechanical vibrations. Journal of King Saud University - Science, 31 (1) (2019), 66-74.

[20] R.B. Thompson, Physical Principles of Measurements with EMAT Transducers, in *Physical Acoustics*, W.P. Mason and R.N. Thurston, Eds., Academic Press, New York, USA (1990), Volume 19, pp. 157-200.

[21] M. Hirao and H. Ogi, *EMATs for Science and Industry: Non-contacting Ultrasonic Measurements*, Kluwer Academic, Boston, USA (2003).

[22] R. Ribichini, F. Cegla, P.B. Nagy, P. Cawley, Quantitative modeling of the transduction of electromagnetic acoustic transducers operating on ferromagnetic media. The IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, 57 (12) (2010), 2808-2817.

[23] A. Füzfa, How current loops and solenoids curve space-time. Physical Review D, 93 (2) (2016), 024014.

[24] A.A. Zakharenko, The problem of finding of eigenvectors for 4P-SH-SAW propagation in 6 *mm* media. Canadian Journal of Pure and Applied Sciences, 11 (1) (2017), 4103-4119; DOI: https://doi.org/10.5281/zenodo.1301202.

[25] E. Dieulesaint and D. Royer, *Elastic Waves in Solids: Applications to Signal Processing*, (translated by A. Bastin and M. Motz, Chichester [English]), J. Wiley, New York (1980), 511 pages.

[26] A.A. Zakharenko, On new interfacial four-potential acoustic SH-wave in dissimilar media pertaining to transversely isotropic class 6 *mm*. Canadian Journal of Pure and Applied Sciences, 11 (3) (2017), 4321-4328; DOI: https://doi.org/10.5281/zenodo.1301215.

[27] A.A. Zakharenko, On existence of new dispersive four-potential SH-waves in 6 *mm* plates for new communication era based on gravitational phenomena. Canadian Journal of Pure and Applied Sciences, 12 (3) (2018), 4585-4591; DOI: http://doi.org/10.5281/zenodo.1471100.

[28] S.I. Khmelnik, *Gravitomagnetism: Nature's Phenomena, Experiments, Mathematical Models*, published by "MiC" - Mathematics in Computer Comp., Printed in the United States of America, Lulu Inc. (2017), 257 pages; DOI: http://doi.org/10.5281/zenodo.1403669.

[29] B.A. Auld, *Acoustic Fields and Waves in Solids*, Volumes I and II (set of two volumes), 2nd edition, Krieger Publishing Company (1990), 878 pages.

[30] Yu.V. Gulyaev, Review of shear surface acoustic waves in solids. The IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, 45 (4) (1998), 935-938.

[31] A.A. Zakharenko, On separation of exchange terms for four-potential acoustic SH-wave case with dependence on gravitational parameters. Hadronic Journal, 41 (4) (2018), 349-370; DOI: http://doi.org/10.5281/zenodo.2842082.

[32] A.A. Zakharenko, Some problems of finding of eigenvalues and eigenvectors for SH-wave propagation in transversely isotropic piezoelectromagnetics. Canadian Journal of Pure and Applied Sciences, 8 (1) (2014), 2783-2787; DOI: https://doi.org/10.5281/zenodo.1300213.

[33] A.A. Zakharenko, Peculiarities study of acoustic waves' propagation in piezoelectromagnetic (composite) materials. Canadian Journal of Pure and Applied Sciences, 7 (2) (2013), 2459-2461; DOI: https://doi.org/10.5281/zenodo.1300926.

[34] Ü. Özgür, Ya. Alivov and H. Morkoç, Microwave ferrites, part 2: Passive components and electrical tuning. Journal of Materials Science: Materials in Electronics, 20 (10) (2009), 911-952.

[35] M. Fiebig, Revival of the magnetoelectric effect. Journal of Physics D: Applied Physics, 38 (8) (2005), R123-R152.

ALGEBRAS GROUPS AND GEOMETRIES 35 319 -327 (2018) - 319 -
DOI 10.29083/AGG.35.03.2018

METRIC TENSOR IS GROUPOID

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Received December 20, 2019

Abstract

Metric tensor is defined as an isomorphism from a module of vector fields (from ring derivations) to a module of differential forms. In fact metric tensor is Grassmann algebra isomorphism, because there are two Grassmann algebras, Grassmann algebra of contra-variant tensors and Grassmann algebra of covariant tensors. Thus metric tensor is a groupoid category. The Minkowski metric can be GL-lifted, to arbitrary material medium and to arbitrary metric, and this GL-lift unify the special and general relativities [Santilli 2010, 2015]. The Lorentz transformations, metric tensor automorphism group, hold well for arbitrary metric tensor, and thus for arbitrary Christoffel-Riemann curvature tensor.

Keywords: GL-lift of Minkowski metric tensor, Lorentz group for arbitrary metric tensor, groupoid category

2000 Mathematics Subject Classification. 51B20 Minkowski geometry, 53B30 Lorentz metric is groupoid, 83A05 Special relativity, 83Cxx General relativity

Invited paper at the 7th International Conference on Pure Mathematics
http://www.icpam.org July 10-13, 2018 Budapest

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Physics and Astronomy Classification Scheme (PACS) - 2010. 02. Mathematical methods in physics; 03.30.+p Special relativity; 04 General relativity and gravitation http://www.aip.org/pubservs/pacs.html

1 Metric tensor play a double role

- Metric tensor g is measuring distances in affine spacetime.
- Elwin Bruno Christoffel (1829-1900) in 1869 invented the derivation of the tensor algebra, known as the covariant derivation. The Christoffel derivation ∇ commute with the evaluation/contraction. Tensor g is a gravitational potential for the Christoffel derivation, $\nabla g = 0$.

Above interpretations of a metric tensor are *missing* the most important primordial definition of the metric tensor: it is wave - particle duality. Gregorio Ricci of Pisa, discovered before 1900, the most important duality in the Universe, that the Universe consists of two kinds of tensors, the contravariant tensors, and covariant tensors. The contra-variant tensors are generated by ring-derivations and therefore should be associated with particle-face. The covariant tensors are owners of crests, fronts, and wave-longitudes, and therefore must be associated with wave-face of Universe. A metric tensor should be defined as an iso-morphisms from an algebra of contra-variant tensors to an algebra of covariant tensors. Therefore the genuine definition of the metric-tensor is ambient-dependent particle-wave iso-duality.

Metric tensor g is defined as ambient-dependent (refraction-dependent, gravity-dependent) **isomorphism** of tensor-algebras. It is particle-wave duality. And this most important duality of the Universe was discovered by Italian mathematician Gregorio Ricci-Curbastro (1853-1925) at Padua University before 1900. Posteriorly this duality in less clarity was postulated by Duc Louis De Broglie in his Doctoral Thesis in 1924, and another version of this duality was given by Arthur Compton in 1923.

1.1 Definition (Metric tensor is a groupoid). Let $\mathcal{F}$ denotes an algebra of scalar fields. A metric tensor g is defined as an $\mathcal{F}$-module isomorphism from a $\mathcal{F}$-module of Leibniz derivations, $\operatorname{der}\mathcal{F}$, to $\mathcal{F}$-module of differential forms, giving rise to gròupoid category, Figure 1. If $M \in \operatorname{der}\mathcal{F}$ is a Leibniz derivation, then gM denotes a g-companion medium-dependent differential

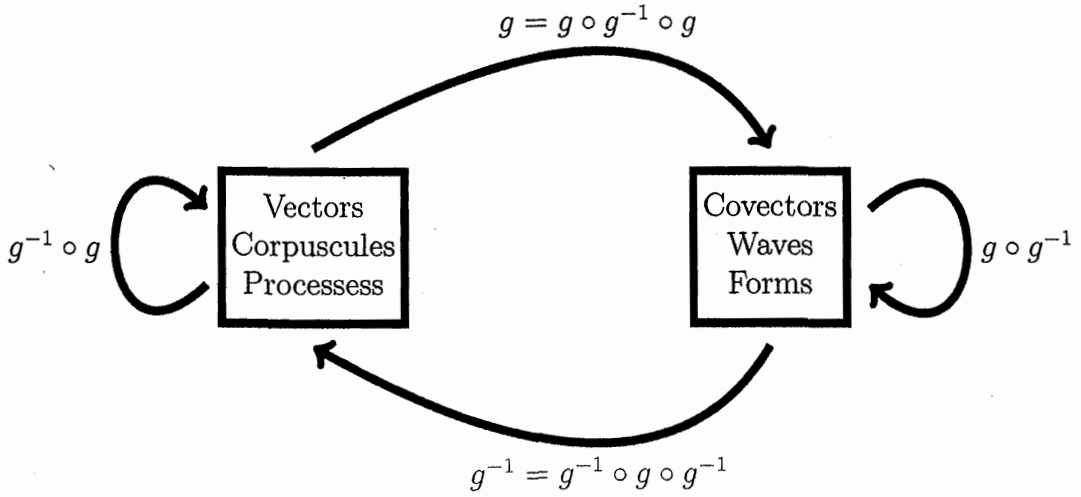
form. A scalar field, $(gM)N$, is the evaluation of a differential form gM with a vector field N. Within groupoid as opposed to the group: $g^{-1}\circ g \quad \neq \quad g\circ g^{-1}$.

Split or decomposition of a metric tensor. Let $M \in \operatorname{der}\mathcal{F}$ be non-isotropic vector field, then

$$g = \frac{(gM)\otimes(gM)}{(gM)M} + g_M, \qquad g_M \in \operatorname{iso}\{\ker(gM), \ker M\}.$$
$$g^{-1} = \frac{M\otimes M}{(gM)M} + (g_M)^{-1}, \qquad (g_M)^{-1} \in \operatorname{iso}\{\ker M, \ker(gM)\}.$$

$$g\circ g^{-1} = \frac{(gM)\otimes M}{(gM)M} + g_M \circ (g_M)^{-1}$$
$$\neq \quad \frac{M\otimes(gM)}{(gM)M} + (g_M)^{-1}\circ g_M = g^{-1}\circ g.$$

Figure 1: A metric tensor is a groupoid categoty. This H. Brandt groupoid consists of four categorical arrows: $\{g, \quad g^{-1}, \quad g^{-1}\circ g, \quad g\circ g^{-1}\}$.



2 Santilli's conceptual unification of general relativity with special relativity

The meaning of the adjective 'relativity' within the *general relativity* is empty, because no concept is declared to be *relative*, *i.e.* the reference dependent, within such gravity theory. The lack of *relativity concepts* in the presence of gravity have been seen in the absence of analogy of the Lorentz transformations being a fundamental concept of special relativity in the absence of the gravity, and in the absence of a material medium.

One of the most important conceptual and philosophical problem was the lack of compatibility of non-vanishing curvature and presence of material medium, with the special relativity of vacuum and absence of gravity [Santilli 2010; 2015]. It is wide spread wrong belief (a strong preconception) that the necessary and sufficient condition on metric **tensor** that assure the existence of the Lorentz transformations (as the symmetry of this metric **tensor**) must be vanishing of the Christoffel curvature.

$$\text{morphism} \quad g \quad \xrightarrow{GL-\text{action/lift}} \quad U^{*-1} \circ g \circ U^{-1} \quad \left\{ \begin{array}{l} \text{new lifted} \\ \text{morphism} \end{array} \right. \tag{2.1}$$

2.1 Definition (Deformation tensor). Let $U \in GL$. In elasticity theory the GL-dependent tensor, $U^{*-1} \circ g \circ U^{-1}$, is said to be the transformed geometry, and the difference, $(U^{*-1} \circ g \circ U^{-1}) - g$, is said to be the deformation tensor.

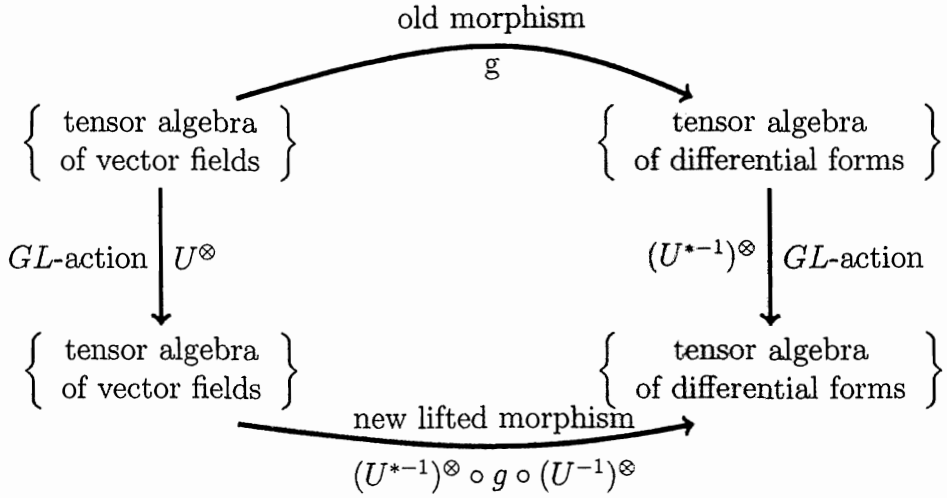
2.2 Clarification (Lifted geometry). Santilli resolved the lack of compatibility of general relativity with special relativity by observing that the Minkowski metric η can be GL-lifted to arbitrary metric g within arbitrary dense medium and arbitrary curvature with arbitrary gravity field [Santilli 2010, 2015], Figure 2,

$$\eta \quad \xrightarrow{U \in GL\text{-lift}} \quad g = U^{*-1} \circ \eta \circ U^{-1} \tag{2.2}$$

Mendel Sachs re-named the GL-group as the Einstein group of general relativity [Sachs 1982].

The Santilli lift (2.2) gives the unification of special and general relativities. The Santilli GL-lift give rise to isotopy maps of all concomitant tensors depending on metric tensor. Isotopy is not GL-covariance because is acting only on metric tensor, leaving all vectors and other tensors intact. For

Figure 2: $GL(\mathcal{F})$-action (non-canonical or non-unitary lift) on **morphism** between tensors. A group action $U \in GL(\mathcal{F})$ and pull-back (transposition) U^* **cannot** be composed. This GL-action was proposed by Santilli as unifying special relativity of vacuum (of absence of gravity and absence of material dense medium) with general relativity of arbitrary material environment (medium) and of arbitrary gravity field.



example, for $X \in \mathrm{der}\,\mathcal{F}$,

$$(\eta X)X \xrightarrow{\text{isotopy}} (gX)X = \{(U^{-1*} \circ \eta \circ U^{-1})X\}X = (\eta U^{-1}X)(U^{-1}X) \quad (2.3)$$

For a sets of differential forms $\{\alpha^\mu\}$ and $\{\beta_\mu\}$, that do not need to be related to Cartan's concept of the moving frames because the metric tensor, as every tensor, is frame-free, co-frame-free, - one can present the metric **tensor** as the sum, with $GL(\mathcal{F})$-action for $U \in GL(\mathcal{F})$,

$$g = \sum \alpha^\mu \otimes \beta_\mu \xrightarrow{GL\text{-action};\quad GL\text{-lift}} (U^{*-1} \otimes U^{*-1})\, g \quad (2.4)$$

The map from the tensor field g (2.4) - to $\mathcal{F}$-module **morphism** $\bar{g}$ from $\mathrm{der}\,\mathcal{F}$ to $(\mathrm{der}\,\mathcal{F})^*$, - is the composition of the annihilation operator represented by the evaluation, with creation operator, $\wedge$ or $\otimes$,

$$g = \alpha \otimes_{\mathcal{F}} \beta + \ldots \quad \longleftrightarrow \quad \bar{g} = \wedge_\alpha \circ \mathrm{ev}_\beta + \ldots \quad (2.5)$$

The above morphism $\bar{g}$ extends to algebra map. The **morphism** $\bar{g}$ between tensors possesses compatible $GL(\mathcal{F})$-action,

$$\bar{g} \quad \xrightarrow{GL-\text{action}} \quad U^{*-1} \circ \bar{g} \circ U^{-1} \tag{2.6}$$

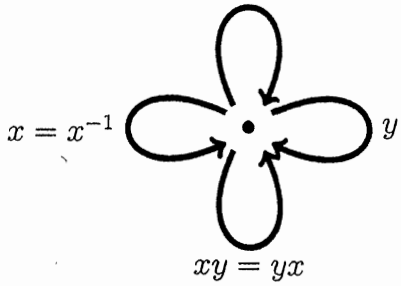
One can re-derive the $GL(\mathcal{F})$-action (2.6) starting from the correspondence (2.5). The evaluation ev_β is a $\mathcal{F}$-module morphism from $\mathrm{der}\,\mathcal{F}$ to $\mathcal{F}$. Whereas the creation $\wedge_\alpha$ is a $\mathcal{F}$-module morphism from $\mathcal{F}$ to $(\mathrm{der}\,\mathcal{F})^*$. This leads to the following $GL(\mathcal{F})$-actions compatible with (2.6),

$$\begin{aligned}
\mathrm{ev}_\beta \quad &\xrightarrow{GL-\text{action}} \quad \mathrm{ev}_\beta \circ U^{-1} = \mathrm{ev}_{U^{-1*}\beta} \\
\wedge_\alpha \quad &\xrightarrow{GL-\text{action}} \quad U^{*-1} \circ \wedge_\alpha = \wedge_{U^{-1*}\alpha} \circ U^{*-1} \\
\wedge_\alpha \circ \mathrm{ev}_\beta \quad &\xrightarrow{GL-\text{action}} \quad U^{*-1} \circ \wedge_\alpha \circ \mathrm{ev}_\beta \circ U^{-1}
\end{aligned} \tag{2.7}$$

2.3 Side remark (Pedagogical example). An illustrative example of four-element group (with unique unit) versus four-element groupoid (with **two** units).

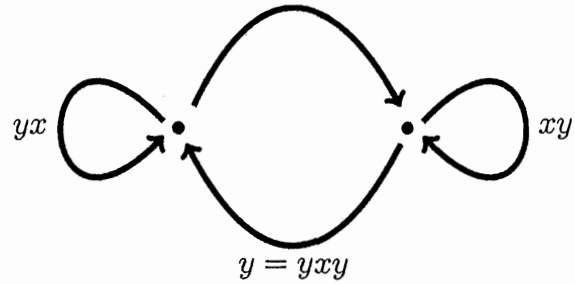
Felix Klein Group $\mathbb{Z}_2 \times \mathbb{Z}_2$

$x^2 = y^2 = 1$



Heinrich Brandt 1927
Groupoid is not a group

$x = xyx$



GROUP

	1	x	y	xy
$x^2 \equiv 1$	1	x	y	xy
x	x	1	xy	y
y	y	yx	1	x
xy	xy	y	x	1

GROUPOID

	xy	x	y	yx
xy	xy	-	y	-
x	x	-	yx	-
y	-	xy	-	y
yx	-	x	-	yx

2.4 Definition (Reference material body). A material body *at rest* is said to be the **reference** body. To be at rest is tautology to possess the **zero** velocity. The relative concepts depend on a choice of a body at rest, depend on a choice of a **reference** body.

Relative concepts, like a relative velocity and an electric field, were well known in a pre-relativity era. Hermann Minkowski introduced new **absolute** concepts, the absolute concept of a time-like body (as an iso-position process preserving positions), the absolute concept of an event as an element of affine spacetime, absolute electromagnetic field, absolute energy-momenta stress-tensor, etc. An absolute affine space-time of events was independently also advocated by Alfred North Whitehead (1861-1947).

A time-like body is reference-free, basis-free, frame-free and coordinate-free, and as well an event is reference-free and coordinate-free. For this reason the relativity theory should be renamed as the theory of absoluteness.

The philosophical essence of the colloquial relativity theory is a dichotomy:

$$\text{Dichotomy:} \qquad \textbf{absolute} \longleftrightarrow \textbf{relative}.$$

2.5 Example (Irrelevant dual frames). The basis-free and coordinate-free metric **tensor** is confusingly and misleadingly 'defined' in terms of irrelevant coordinates, coordinates that are mortal for a concept of a metric **tensor**. Let X be an algebra derivation,

$$g_{\mu\nu} \equiv (g\partial_\mu)\partial_\nu = g(\partial_\mu \otimes \partial_\nu) = \mathrm{ev}(g \otimes \partial_\mu \otimes \partial_\nu), \tag{2.8}$$

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \quad \Longrightarrow \quad gX = (g_{\mu\nu}\, dx^\mu \otimes dx^\nu)X = (g_{\mu\nu} X x^\nu) dx^\mu. \tag{2.9}$$

In (2.9) $\{dx^\mu\}$ are differential forms, irrelevant co-frames, that in some textbooks are interpreted as the **scalar** infinitesimal displacements that leads to 'old-fashioned' misleading interpretation of a metric **tensor** as an infinitesimal **scalar** distance squared denoted by ds^2. Such innocent notation is the source of mis-understandings, because does not exists neither a scalar 's' nor the differential ds, and $g \neq ds \otimes ds$. The Riemannian metric (2.9) is the genuine $GL(\mathcal{F})$-tensor.

References

Sachs Mendel, General Relativity and Matter. Reidel Publishing Co. Dordrecht 1982

Santilli Ruggero Maria. International Journal of Modern Physics **D 7** (1998) 351

Santilli Ruggero Maria, Isodual Theory of Antimatter with applications to Antigravity, Grand Unification and Cosmology. Fundamental Theories of Physics Volume 151. Springer 2006

Santilli Ruggero Maria, Experimental verifications of isoredshift with possible absence of universe expansion, big bang, dark matter, and dark energy. The Open Astronomy Journal **3** (2010) 126 - 144

Santilli Ruggero Maria, Compatibility of arbitrary speeds with special relativity axioms for interior dynamical problems. American Journal of Modern Physics 2015, pages 1-19

Santilli Ruggero Maria, Representation of galactic dynamics via isoshifts without universe expansion, dark matter and dark energy. American Journal of Modern Physics Volume 4 No. 2 (February 25) (2015) pages 26-43, doi: 10.11648/j.ajmp.20150402.11

Santilli Ruggero Maria, Outline of the new era in mathematics and in its applications to new sustainable energies. Santilli International Symposium on Sustainable Mathematics Applications. November 2016. Hainan-Island, China

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